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Drinfel'd–Jimbo coproduct of quantized GIM Lie algebras

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Abstract

We define a sequence of quotient algebras of a quantized GIM Lie algebra's tensor algebras, modulo ideals generated by some imaginary root vectors, so that there are algebra morphisms similar to the Drinfel'd–Jimbo coproduct of quantum groups, and hence the quantized GIM Lie algebra has properties which are similar to those of Hopf algebras. Weight modules and quantum loop modules of the corresponding quantum affine Kac–Moody algebra are used to construct tensor modules of the quantized GIM Lie algebra via the coproduct.

Keywords: Generalized intersection matrices; Quantum groups; Hopf algebras

1. Introduction

One of the most important properties of quantum groups of Drinfel'd–Jimbo is that they have Hopf algebra structure, which plays a crucial role in their representation theory. Results on tensor product of modules of classical semisimple Lie algebras have been extended to quantum groups via Drinfel'd–Jimbo coproduct. In recent years generalization of quantum groups such as quantum affinizations attracts more and more attention (cf. [4] and references therein).

In this paper we continue to study quantized GIM Lie algebras U_q determined by the 2-affinization $(a_{ij})_{1\leqslant i,j\leqslant \ell+2}$ of a finite Cartan matrix $(a_{ij})_{1\leqslant i,j\leqslant \ell}$ of ADE type, which is an important example of generalized intersection matrices due to Saito [11] and Slodowy [12]. Although

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the root system of U_q admits a natural Weyl group action [10] and hence U_q has Lusztig symmetries [14], U_q is not a Hopf algebra with Drinfel'd–Jimbo coproduct in an essential way: U_q has other imaginary root vectors than usual quantum groups. However, similar to the antipode and counit of a Hopf algebra, U_q does have an anti-automorphism S and an algebra morphism $\varepsilon : \mathrm{U}_q \to \mathbb{Q}(q)$ which have a close relation with Drinfel'd–Jimbo coproduct. On the other hand, to describe the tensor subcategory of U_q is a fundamental problem in representation theory which requires a coproduct structure. As kindly pointed out by the referee (cf. also [4]), in the case of quantum loop algebras of Kac–Moody algebras, the Drinfel'd–Jimbo coproduct is defined on the loop-like generators of Drinfel'd type via completion process. In the case of quantized GIM Lie algebras, the presentation in terms of Drinfel'd generators is not available at present. A clue in this direction may be quantum toroidal Lie algebras introduced in [3], which are quotient algebras of quantized GIM Lie algebras, as toroidal Lie algebras are quotient algebras of GIM Lie algebras modulo some highly non-trivial ideals [1].

Here we adopt the elementary way of making Drinfel'd–Jimbo coproduct to be an algebra morphism by considering a suitable quotient algebra $\tilde{\otimes}^2 \mathrm{U}_q$ of the tensor algebra $\mathrm{U}_q \otimes \mathrm{U}_q$. It remains to establish coassociativity of Drinfel'd–Jimbo coproduct in some sense. To this end we first generalize coassociativity as in Definition 3.1, then define for each $n \geqslant 2$ a quotient algebra $\tilde{\otimes}^n \mathrm{U}_q$ of an n-fold tensor product of U_q with a fixed way of inserting parentheses, such that there is a well-defined algebra morphism $\triangle_{n-1} : \mathrm{U}_q \to \tilde{\otimes}^n \mathrm{U}_q$ given by Drinfel'd–Jimbo coproduct (cf. Proposition 3.1). We emphasize that this notation is not the completion $\hat{\otimes}^n \mathrm{U}_q$ of the tensor product $\otimes^n \mathrm{U}_q$. These quotient algebras are obtained by modulo ideals determined by some imaginary root vectors. Our computation shows that such a construction admits a structure on U_q with Drinfel'd–Jimbo coproduct similar to Hopf algebras, and some general identities of Hopf algebras still hold in this construction (cf. Propositions 2.1, 2.2, 3.2, 3.3 below).

Thus there does exist a subcategory of U_q-modules closed under tensor product via the algebra morphisms $\triangle_{n-1}$. At the moment such a subcategory is not well studied. However, we shall show that it contains the category of all weight $\mathrm{U}_q(\mathfrak{g})$-modules, where $\mathfrak{g}$ is the affine Kac–Moody Lie algebra determined by the affine Cartan matrix $(a_{ij})_{1 \leqslant i,j \leqslant \ell+1}$. Indeed, similar to the setting of quantum toroidal algebras [3], $\mathrm{U}_q(\mathfrak{g})$ can be embedded into U_q in an obvious way, and there is also an epimorphism from U_q to $\mathrm{U}_q(\mathfrak{g})$. More precisely, for any collection of weight $\mathrm{U}_q(\mathfrak{g})$-modules $V_1, \ldots, V_n$, $V_1 \otimes \cdots \otimes V_n$ is a tensor module induced by the usual Drinfel'd–Jimbo coproduct, and hence a U_q-module induced by the above mentioned epimorphism from U_q to $\mathrm{U}_q(\mathfrak{g})$. As an application of the coproduct construction, we show that $V_1 \otimes \cdots \otimes V_n$ is in fact a $\tilde{\otimes}^n \mathrm{U}_q$-module, which implies that $V_1 \otimes \cdots \otimes V_n$ is a tensor U_q-module induced by the coproduct $\triangle_{n-1} : \mathrm{U}_q \to \tilde{\otimes}^n \mathrm{U}_q$.

During the computation, we find that, associated to a $\mathrm{U}_q(\mathfrak{g})$-module V, the underlying space of Chari–Greenstein's quantum loop module $L(V)$ (cf. [2]) also inherits a U_q-module structure. Moreover, for any collection of weight $\mathrm{U}_q(\mathfrak{g})$-modules $V_1, \ldots, V_n$ we show that the quantum loop module $L(V_1 \otimes \cdots \otimes V_n)$ is also a $\tilde{\otimes}^n \mathrm{U}_q$-module, and hence a tensor U_q-module induced by the coproduct. These new modules are expected to be helpful for the study of infinite-dimensional U_q-modules via the representation theory of classical quantum affine Kac–Moody algebras.

In Section 2 we define the quotient algebra $\tilde{\otimes}^2 \mathrm{U}_q$ so that Drinfel'd–Jimbo coproduct is an algebra morphism, and describe the antipode, counit of U_q to show that U_q is similar to a Hopf algebra. In Section 3 we establish coassociativity by defining a sequence of quotient algebras of tensor algebras of U_q, and prove some identities on higher order coproducts. In Section 4 we construct tensor U_q-modules using weight $\mathrm{U}_q(\mathfrak{g})$-modules and quantum loop modules.

2. Coproduct, counit and antipode

Let $A = (a_{ij})_{1\leqslant i,j\leqslant \ell+2}$ be the 2-affinization of a finite Cartan matrix $\dot{A} = (a_{ij})_{1\leqslant i,j\leqslant \ell}$ of ADE type, cf. [1]. Thus, $a_{\ell+i,\ell+j} = 2$ for $i, j = 2$, the first ℓ principal minor is a finite Cartan matrix while the $(\ell+1)$th is an affine generalized Cartan matrix. Note that the ranks of A and $\dot{A}$ are ℓ. The matrices A obtained this way is an important example of Slodowy's generalized intersection matrix (GIM for short), which has been firstly studied by P. Slodowy [12] and K. Saito [11]. They are closely related to the 2-toroidal Lie algebras, the universal central extensions of double loop algebras $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \otimes \dot{\mathfrak{g}}$, where $\dot{\mathfrak{g}}$ is a complex simple Lie algebra [9].

Let $(\mathcal{H}, \Pi, \Pi^\vee)$ be a realization of A in the sense of Kac [6], i.e., $\mathcal{H}$ is a complex space of dimension $\ell+4$, $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{\ell+2}\} \subset \mathcal{H}^*$ (respectively $\Pi^\vee = \{h_1, h_2, \ldots, h_{\ell+2}\} \subset \mathcal{H}$) is a linearly independent subset of $\mathcal{H}^*$ (respectively $\mathcal{H}$) such that $\alpha_i(h_j) = a_{ji}$ for $1 \leqslant i, j \leqslant \ell+2$. As usual we define the following lattice

$$\Gamma = \bigoplus_{i=1}^{\ell+2} \mathbb{Z}\alpha_i \subset \mathcal{H}^*. \tag{2.1}$$

Define $\Lambda_1, \Lambda_2 \in \mathcal{H}^*$ by $\Lambda_i(h_j) = 0$ for $i = 1, 2$, $1 \leqslant j \leqslant \ell$ and $\Lambda_i(h_{\ell+j}) = \delta_{ij}$ for $i, j = 1, 2$. It follows that $\alpha_1, \ldots, \alpha_{\ell+2}, \Lambda_1, \Lambda_2$ form a basis of $\mathcal{H}^*$. Similarly, define $d_1, d_2 \in \mathcal{H}$ by $\alpha_i(d_j) = 0$ for $j = 1, 2$, $1 \leqslant i \leqslant \ell$ and $\alpha_{\ell+i}(d_j) = \delta_{ij}$ for $i, j = 1, 2$. Then $h_1, \ldots, h_{\ell+2}, d_1, d_2$ form a basis of $\mathcal{H}$. Introduce $\delta_1, \delta_2 \in \mathcal{H}^*$ such that $\alpha_{\ell+i} = \delta_i - \theta$ for $i = 1, 2$, where θ is the unique highest root of $\dot{\mathfrak{g}}$. Then it is easy to see that $\delta_i(d_j) = \delta_{ij}$ since $\theta(d_j) = 0$. Also introduce $c_1, c_2 \in \mathcal{H}$ such that $h_{\ell+i} = c_i - \theta^\vee$ for $i = 1, 2$. It is easy to see that $\Lambda_i(c_j) = \delta_{ij}$ since $\Lambda_i(\theta^\vee) = 0$. Hence there is a symmetric bilinear non-degenerate form $(.\mid.)$ on $\mathcal{H}^*$ such that, for $1 \leqslant i, j \leqslant \ell$ and $k, k' = 1, 2$,

$$(\alpha_i \mid \alpha_j) = a_{ij}, \qquad (\alpha_i \mid \delta_k) = (\alpha_i \mid \Lambda_k) = (\delta_k \mid \delta_{k'}) = (\Lambda_k \mid \Lambda_{k'}) = 0,$$
$$(\delta_k \mid \Lambda_{k'}) = \delta_{kk'}. \tag{2.2}$$

Note that $(\delta_i \mid \alpha_j) = 0$ for $1 \leqslant j \leqslant \ell+2$ and $(\Lambda_k \mid \alpha_{\ell+j}) = \delta_{kj}$ for $k, j = 1, 2$.

Let q be an indeterminate. Then we have the usual notation $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ and $[n]! = [n][n-1]\cdots[2][1]$. Also, for any letter x, denote $x^{(n)} = \frac{x^n}{[n]!}$ (except notation such as $R_{ij}^{(n)}$ in Section 3). Then the quantized GIM Lie algebra U_q associated to the GIM matrix $A = (a_{ij})_{i,j=1}^{\ell+2}$ and the lattice Γ given by (2.1) has the following presentation. Different from [2,14], we omit the derivatives D_i for simplicity. For details see Remark 4.1 below.

Definition 2.1. (See [14].) The algebra $\mathrm{U}_q = \mathrm{U}_q(\mathfrak{g}_{\mathrm{gim}})$ is a unitary associative algebra over $\mathbb{Q}(q)$ with the following presentation.

Generators: K_α $(\alpha \in \Gamma)$, E_j, F_j $(1 \leqslant j \leqslant \ell+2)$, $C_i^{\pm\frac{1}{2}}$ $(i = 1, 2)$.

Relations:

Re.1 $K_\alpha K_\beta = K_{\alpha+\beta},\ K_0 = 1,$

$C_i^{\pm\frac{1}{2}}$ is central and $\left(C_i^{\pm\frac{1}{2}}\right)^2 = K_{\delta_i}^{\pm 1},\ i = 1, 2,$

$K_\alpha E_j K_\alpha^{-1} = q^{\alpha(h_j)} E_j,\ K_\alpha F_j K_\alpha^{-1} = q^{-\alpha(h_j)} F_j,\ 1 \leqslant j \leqslant \ell + 2,$

$[E_j, F_j] = \dfrac{K_j - K_j^{-1}}{q - q^{-1}}$, where $K_j = K_{\alpha_j},\ 1 \leqslant j \leqslant \ell + 2.$

Re.2 For $a_{ij} \leqslant 0$ and $1 \leqslant i \neq j \leqslant \ell + 2$,

$$[E_i, F_j] = 0,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s E_i^{(1-a_{ij}-s)} E_j E_i^{(s)} = 0,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s F_i^{(1-a_{ij}-s)} F_j F_i^{(s)} = 0. \tag{2.3}$$

Re.3 For $a_{ij} = 2,\ i \neq j,$ i.e., $\ell + 1 \leqslant i \neq j \leqslant \ell + 2$,

$$\sum_{s=0}^{3} (-1)^s E_i^{(3-s)} F_j E_i^{(s)} = 0 = \sum_{s=0}^{3} (-1)^s F_i^{(3-s)} E_j F_i^{(s)}, \tag{2.4}$$

$$[E_i, E_j] = [F_i, F_j] = 0. \tag{2.5}$$

Notation. From now on we denote $K_{\sum t_i \alpha_i}$ by $K_{\sum t_i i}$ for any $\sum t_i \alpha_i \in \Gamma$. For example, $K_{3\alpha_i - 2\alpha_j}$ is denoted as K_{3i-2j}. Relations (2.4) and (2.5) are called GIM-Serre relations.

At first we have the following

Lemma 2.1. *There is an anti-automorphism S of* U_q (*as an algebra over* $\mathbb{Q}(q)$) *given by*

$$S(E_j) = -K_j^{-1} E_j, \qquad S(F_j) = -F_j K_j,$$
$$S(K_\alpha) = K_{-\alpha}, \qquad S\left(C_i^{\pm\frac{1}{2}}\right) = C_i^{\mp\frac{1}{2}} \tag{2.6}$$

for $1 \leqslant j \leqslant \ell + 2$, $\alpha \in \Gamma$ *and* $i = 1, 2$.

Proof. We check directly S preserves the relations Re.1, Re.2 and Re.3 to show that $S : \mathrm{U}_q \to \mathrm{U}_q^{\mathrm{opp}}$ is an algebra morphism. The fact that S preserves Re.1 and Re.2 is well known (cf. [8]). Now assume that $a_{ij} = 2$ for $i \neq j$. Then Re.3 is equivalent to

$$E_i^3 F_j - [3] E_i^2 F_j E_i + [3] E_i F_j E_i^2 - F_j E_i^3 = 0;$$
$$F_i^3 E_j - [3] F_i^2 E_j F_i + [3] F_i E_j F_i^2 - E_j F_i^3 = 0;$$
$$[E_i, E_j] = [F_i, F_j] = 0.$$

By a direct computation we have that

$$S(F_j)\big(S(E_i)\big)^3 - [3]S(E_i)S(F_j)\big(S(E_i)\big)^2 + [3]\big(S(E_i)\big)^2 S(F_j)S(E_i) - \big(S(E_i)\big)^3 S(F_j)$$
$$= -q^2 K_{3i}^{-1} K_j \big\{E_i^3 F_j - [3]E_i^2 F_j E_i + [3]E_i F_j E_i^2 - F_j E_i^3\big\} = 0,$$

which means that S preserves the first relation in Re.3. Similarly, S preserves the second relation in Re.3. Also we have that

$$S(E_j)S(E_i) - S(E_i)S(E_j) = q^2 K_i^{-1} K_j^{-1}(E_j E_i - E_i E_j) = 0,$$

which means that S preserves the relation $[E_i, E_j] = 0$. Similarly, S preserves the relation $[F_i, F_j] = 0$. So, $S : \mathrm{U}_q \to \mathrm{U}_q^{\mathrm{opp}}$ is an algebra morphism. It is easy to see that S is invertible with the inverse S' given by

$$S'(E_j) = -E_j K_j^{-1}, \qquad S'(F_j) = -K_j F_j,$$
$$S'(K_\alpha) = K_{-\alpha}, \qquad S'\big(C_i^{\pm\frac{1}{2}}\big) = C_i^{\mp\frac{1}{2}} \tag{2.7}$$

for $1 \leqslant j \leqslant \ell + 2$, $\alpha \in \Gamma$ and $i = 1, 2$. $\square$

It is easy to see that there is an algebra morphism $\varepsilon : \mathrm{U}_q \to \mathbb{Q}(q)$ given by

$$\varepsilon(E_j) = \varepsilon(F_j) = 0, \qquad \varepsilon(K_\alpha) = \varepsilon\big(C_i^{\pm\frac{1}{2}}\big) = 1 \tag{2.8}$$

for $1 \leqslant j \leqslant \ell + 2$, $\alpha \in \Gamma$ and $i = 1, 2$.

As pointed out in [14], and it is easy to see that, the following Drinfel'd–Jimbo coproduct $\triangle$ of quantum groups given by

$$\triangle(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \triangle(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\triangle(K_\alpha) = K_\alpha \otimes K_\alpha, \qquad \triangle\big(C_j^{\pm\frac{1}{2}}\big) = C_j^{\pm\frac{1}{2}} \otimes C_j^{\pm\frac{1}{2}}, \tag{2.9}$$

for $1 \leqslant i \leqslant \ell + 2$, $j = 1, 2$ and $\alpha \in \Gamma$, is not an algebra homomorphism from U_q to $\mathrm{U}_q \otimes \mathrm{U}_q$. To define a quotient algebra of $\mathrm{U}_q \otimes \mathrm{U}_q$ as follows, we list the following formulas in $\mathrm{U}_q \otimes \mathrm{U}_q$, which can be verified by a long but direct computation,

$$\sum_{s=0}^{3}(-1)^s\big(\triangle(E_i)\big)^{(3-s)}\triangle(F_j)\big(\triangle(E_i)\big)^{(s)}$$
$$= \frac{(q^{-2}-1)}{[2]}\big\{K_{2i}E_i \otimes \big[E_i^2 F_j - \big(1+q^{-2}\big)E_i F_j E_i + q^{-2}F_j E_i^2\big]$$
$$- K_i\big[E_i^2 F_j - \big(1+q^2\big)E_i F_j E_i + q^2 F_j E_i^2\big] \otimes K_j^{-1}E_i\big\} \neq 0; \tag{2.10}$$

$$\sum_{s=0}^{3}(-1)^s\big(\triangle(F_i)\big)^{(3-s)}\triangle(E_j)\big(\triangle(F_i)\big)^{(s)}$$
$$=\frac{(q^{-2}-1)}{[2]}\big\{\big[F_i^2E_j-\big(1+q^{-2}\big)F_iE_jF_i+q^{-2}E_jF_i^2\big]\otimes K_{2i}^{-1}F_i$$
$$-K_jF_i\otimes K_i^{-1}\big[F_i^2E_j-\big(1+q^2\big)F_iE_jF_i+q^2E_jF_i^2\big]\big\}\neq 0; \tag{2.11}$$
$$\triangle(E_i)\triangle(E_j)-\triangle(E_j)\triangle(E_i)=\big(q^{-2}-1\big)\{K_jE_i\otimes E_j-K_iE_j\otimes E_i\}\neq 0; \tag{2.12}$$
$$\triangle(F_i)\triangle(F_j)-\triangle(F_j)\triangle(F_i)=\big(q^{-2}-1\big)\big\{F_j\otimes K_j^{-1}F_i-F_i\otimes K_i^{-1}F_j\big\}\neq 0. \tag{2.13}$$

We may simplify notations in (2.10)–(2.13) for further considerations as follows. At first, there is an automorphism ω of U_q over $\mathbb{Q}(q)$ given by:

$$K_\alpha\mapsto K_{-\alpha},\quad C_i^{\pm\frac{1}{2}}\mapsto C_i^{\mp\frac{1}{2}},\quad E_j\mapsto F_j,\quad F_j\mapsto E_j \tag{2.14}$$

for $\alpha\in\Gamma$, $i=1,2$ and $1\leqslant j\leqslant \ell+2$. Thus we have an automorphism of $\mathrm{U}_q\otimes\mathrm{U}_q$ denoted as:

$$\hat{\otimes}^2\omega:=\tau(\omega\otimes\omega), \tag{2.15}$$

where $\tau:\mathrm{U}_q\otimes\mathrm{U}_q\to\mathrm{U}_q\otimes\mathrm{U}_q$ is the flip map. Also, by the definition of Lusztig symmetries T_i of U_q (cf. Theorem 2.1 in [14]), we set, for $\ell+1\leqslant i\neq j\leqslant \ell+2$,

$$\hat{T}_i(F_j)=[2]T_i(F_j)=E_i^2F_j-\big(1+q^{-2}\big)E_iF_jE_i+q^{-2}F_jE_i^2;$$
$$\hat{T}_i'(F_j)=[2]q^2T_i'(F_j)=E_i^2F_j-\big(1+q^2\big)E_iF_jE_i+q^2F_jE_i^2. \tag{2.16}$$

Thus (2.10) becomes

$$\sum_{s=0}^{3}(-1)^s\big(\triangle(E_i)\big)^{(3-s)}\triangle(F_j)\big(\triangle(E_i)\big)^{(s)}$$
$$=\frac{(q^{-2}-1)}{[2]}\big\{K_{2i}E_i\otimes\hat{T}_i(F_j)-K_i\hat{T}_i'(F_j)\otimes K_j^{-1}E_i\big\}, \tag{2.17}$$

and (2.11) is obtained by applying $\hat{\otimes}^2\omega$ to (2.17), while (2.13) is obtained by applying $\hat{\otimes}^2\omega$ to (2.12). (Note that $(\hat{\otimes}^2w)\triangle(E_j)=\triangle(F_j)$ and $(\hat{\otimes}^2w)\triangle(F_j)=\triangle(E_j)$ for $1\leqslant j\leqslant\ell+2$.)

For $a_{ij}=2$ and $i\neq j$ we define the following elements in $\mathrm{U}_q\otimes\mathrm{U}_q$:

$$r_{ij}=K_{2i}E_i\otimes\hat{T}_i(F_j)-K_i\hat{T}_i'(F_j)\otimes K_j^{-1}E_i, \tag{2.18}$$
$$s_{ij}=K_jE_i\otimes E_j-K_iE_j\otimes E_i, \tag{2.19}$$

and define the following quotient algebra

$$\tilde{\otimes}^2\mathrm{U}_q=(\mathrm{U}_q\otimes\mathrm{U}_q)/\big\langle r_{ij},\ s_{ij},\ \big(\hat{\otimes}^2\omega\big)(r_{ij}),\ \big(\hat{\otimes}^2\omega\big)(s_{ij}):\ \ell+1\leqslant i\neq j\leqslant\ell+2\big\rangle, \tag{2.20}$$

where $\hat{\otimes}^2\omega$ is given by (2.15). Since U_q is $\mathbb{Z}\Gamma$-graded by assigning

$$\deg(K_\alpha)=\deg\bigl(C_i^{\pm\frac{1}{2}}\bigr)=0, \qquad \deg(E_j)=\alpha_j=-\deg(F_j),$$

it follows that $\mathrm{U}_q\otimes\mathrm{U}_q$ is $\mathbb{Z}\Gamma\times\mathbb{Z}\Gamma$-graded. Also, since r_{ij}, s_{ij}, $(\hat{\otimes}^2\omega)(r_{ij})$, $(\hat{\otimes}^2\omega)(s_{ij})$ are homogeneous, the algebra $\tilde{\otimes}^2\mathrm{U}_q$ is also $\mathbb{Z}\Gamma\times\mathbb{Z}\Gamma$-graded. It follows that the two natural embeddings of $\mathrm{U}_q\hookrightarrow\mathrm{U}_q\otimes\mathrm{U}_q$ given by $u\mapsto u\otimes 1$ and $u\mapsto 1\otimes u$ respectively induce two embeddings of $\mathrm{U}_q\hookrightarrow\tilde{\otimes}^2\mathrm{U}_q$. Since, as is well known, the Drinfel'd–Jimbo coproduct $\triangle$ given by (2.9) preserves defining relations Re.1 and Re.2 (cf. (2.3)) of U_q, by the definition of $\tilde{\otimes}^2\mathrm{U}_q$ we obtain the following

Lemma 2.2. *There is a unique algebra morphism $\triangle:\mathrm{U}_q\to\tilde{\otimes}^2\mathrm{U}_q$ satisfying* (2.9).

For the algebra morphism ε given by (2.8), it is easy to see that

$$(1\otimes\varepsilon)(r_{ij})=(1\otimes\varepsilon)\bigl(\bigl(\hat{\otimes}^2\omega\bigr)(r_{ij})\bigr)=(1\otimes\varepsilon)(s_{ij})=(1\otimes\varepsilon)\bigl(\bigl(\hat{\otimes}^2\omega\bigr)(s_{ij})\bigr)=0,$$
$$(\varepsilon\otimes 1)(r_{ij})=(\varepsilon\otimes 1)\bigl(\bigl(\hat{\otimes}^2\omega\bigr)(r_{ij})\bigr)=(\varepsilon\otimes 1)(s_{ij})=(\varepsilon\otimes 1)\bigl(\bigl(\hat{\otimes}^2\omega\bigr)(s_{ij})\bigr)=0.$$

It follows that there are induced algebra morphisms

$$1\otimes\varepsilon:\tilde{\otimes}^2\mathrm{U}_q\to\mathrm{U}_q\otimes\mathbb{Q}(q)\quad\text{and}\quad \varepsilon\otimes 1:\tilde{\otimes}^2\mathrm{U}_q\to\mathbb{Q}(q)\otimes\mathrm{U}_q.$$

Moreover, we have the following

Lemma 2.3. *As $\mathbb{Q}(q)$-algebra morphisms it holds that*

$$(1\otimes\varepsilon)\triangle=\mathbf{can}:\mathrm{U}_q\to\mathrm{U}_q\otimes\mathbb{Q}(q);\qquad (\varepsilon\otimes 1)\triangle=\mathbf{can}:\mathrm{U}_q\to\mathbb{Q}(q)\otimes\mathrm{U}_q. \tag{2.21}$$

Proof. It suffices to check it on generators of U_q. $\square$

Let $\mathbf{m}:\mathrm{U}_q\times\mathrm{U}_q\to\mathrm{U}_q$ be the multiplication of U_q. For the anti-automorphism S of U_q given by (2.6), by a direct computation and using the GIM-Serre relations (2.4), (2.5), we have following identities in $\mathrm{U}_q\otimes\mathrm{U}_q$:

$$\begin{aligned}
\mathbf{m}(1\otimes S)(r_{ij})&=-q^4\bigl\{E_i^3F_j-[3]E_i^2F_jE_i+[3]E_iF_jE_i^2-F_jE_i^3\bigr\}K_j\\
&=0;\\
\mathbf{m}(1\otimes S)\bigl(\bigl(\hat{\otimes}^2\omega\bigr)(r_{ij})\bigr)&=q^{-2}\bigl\{F_i^3E_j-[3]F_i^2E_jF_i+[3]F_iE_jF_i^2-E_jF_i^3\bigr\}K_{3i}\\
&=0;\\
\mathbf{m}(1\otimes S)(s_{ij})&=-q^2E_iE_j+q^2E_jE_i=0;\\
\mathbf{m}(1\otimes S)\bigl(\bigl(\hat{\otimes}^2\omega\bigr)(s_{ij})\bigr)&=-F_jF_iK_{i+j}+F_iF_jK_{i+j}=0.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
\mathbf{m}(S\otimes 1)(r_{ij}) &= -q^4 K_{3i}^{-1}\left\{E_i^3F_j-[3]E_i^2F_jE_i+[3]E_iF_jE_i^2-F_jE_i^3\right\}\\
&=0;\\
\mathbf{m}(S\otimes 1)\big(\big(\hat{\otimes}^2\omega\big)(r_{ij})\big) &= q^{-2}K_j^{-1}\left\{F_i^3E_j-[3]F_i^2E_jF_i+[3]F_iE_jF_i^2-E_jF_i^3\right\}\\
&=0;\\
\mathbf{m}(S\otimes 1)(s_{ij}) &= q^2K_{i+j}\{E_iE_j-E_jE_i\}=0;\\
\mathbf{m}(S\otimes 1)\big(\big(\hat{\otimes}^2\omega\big)(s_{ij})\big) &= -F_jF_i+F_iF_j=0.
\end{aligned}$$

It follows that $\mathbf{m}(S\otimes 1)\triangle$ and $\mathbf{m}(1\otimes S)\triangle$ preserve all defining relations of U_q. Although $\mathbf{m}(S\otimes 1)$ and $\mathbf{m}(1\otimes S)$ are not algebra morphisms, we have the following

Lemma 2.4. *As algebra endomorphisms of* U_q *it holds that*

$$\mathbf{m}(1\otimes S)\triangle=\iota\circ\varepsilon=\mathbf{m}(S\otimes 1)\triangle, \tag{2.22}$$

where ε *is given by* (2.8), $\iota:\mathbb{Q}(q)\to\mathrm{U}_q$ *is the natural embedding* $\iota(a)=a1$ *for* $a\in\mathbb{Q}(q)$, *and* $\triangle$ *is the Drinfel'd–Jimbo coproduct given by* (2.9).

Proof. By checking it directly we see that (2.22) holds on generators of U_q. So, it remains to show that $\mathbf{m}(1\otimes S)\triangle$ and $\mathbf{m}(S\otimes 1)\triangle$ are algebra morphisms. Since $\iota\circ\varepsilon$ is an algebra morphism, it suffices to check that for any $u,v\in\mathrm{U}_q$ it holds that $\mathbf{m}(1\otimes S)\triangle(uv)=\iota\circ\varepsilon(uv)$ and $\mathbf{m}(S\otimes 1)\triangle(uv)=\iota\circ\varepsilon(uv)$. Since S is an anti-automorphism by Lemma 2.1 and $\triangle$ is an algebra morphism by Lemma 2.2, the result follows by a completely similar argument as in Lemma 3.7 of [5]. □

Note that, since $\mathbf{m}$ is not an algebra morphism from $\mathrm{U}_q\otimes\mathrm{U}_q$ to U_q, it does not induce linear maps $\mathbf{m}(1\otimes S)$ and $\mathbf{m}(S\otimes 1)$ from $\tilde{\otimes}^2\mathrm{U}_q$ to U_q. Neither can $1\otimes S$ and $S\otimes 1$ induce maps from $\tilde{\otimes}^2\mathrm{U}_q$ to $\tilde{\otimes}^2\mathrm{U}_q$.

Up to now we obtain the following

Proposition 2.1. *There are* $\mathbb{Q}(q)$*-algebra morphisms* $\triangle:\mathrm{U}_q\to\tilde{\otimes}^2\mathrm{U}_q$ *given by* (2.9), $\varepsilon:\mathrm{U}_q\to\mathbb{Q}(q)$ *given by* (2.8) *and an anti-automorphism* S *of* U_q *given by* (2.6) *such that* (2.21) *and* (2.22) *hold.*

So, U_q is similar to a Hopf algebra, except a suitable coassociativity undefined. Using language of Hopf algebras, we call $\triangle$ the Drinfel'd–Jimbo coproduct, ε the counit and S the antipode of U_q.

For later computation we need to describe the action of the coproduct $\triangle$ on $\hat{T}_i(F_j)$ and $\hat{T}_i'(F_j)$ defined by Lusztig symmetries (cf. (2.16)). Denote the following imaginary root vectors for $\ell+1\leqslant i\neq j\leqslant\ell+2$ in U_q:

$$\Phi_{ij}=E_iF_j-F_jE_i. \tag{2.23}$$

Then $K_\alpha\Phi_{ij}K_\alpha^{-1}=\Phi_{ij}$ for any $\alpha\in\Gamma$. Moreover, we may write (2.16) as

$$\hat{T}_i(F_j) = E_i \Phi_{ij} - q^{-2} \Phi_{ij} E_i, \qquad \hat{T}_i'(F_j) = E_i \Phi_{ij} - q^2 \Phi_{ij} E_i. \tag{2.24}$$

Then we have the following

Lemma 2.5. *Assume that $a_{ij} = 2$ (that is, $\ell + 1 \leqslant i \neq j \leqslant \ell + 2$). Then*

$$\triangle(\Phi_{ij}) := \triangle(E_i)\triangle(F_j) - \triangle(F_j)\triangle(E_i) = \Phi_{ij} \otimes K_j^{-1} + K_i \otimes \Phi_{ij}; \tag{2.25}$$

$$\begin{aligned} \triangle\big(\hat{T}_i(F_j)\big) &:= \triangle(E_i)\triangle(\Phi_{ij}) - q^{-2}\triangle(\Phi_{ij})\triangle(E_i) \\ &= \hat{T}_i(F_j) \otimes K_j^{-1} + K_{2i} \otimes \hat{T}_i(F_j) + \big(q^2 - q^{-2}\big) K_i \Phi_{ij} \otimes K_j^{-1} E_i; \end{aligned} \tag{2.26}$$

$$\begin{aligned} \triangle\big(\hat{T}_i'(F_j)\big) &:= \triangle(E_i)\triangle(\Phi_{ij}) - q^{2}\triangle(\Phi_{ij})\triangle(E_i) \\ &= \hat{T}_i'(F_j) \otimes K_j^{-1} + K_{2i} \otimes \hat{T}_i'(F_j) - \big(q^2 - q^{-2}\big) K_i E_i \otimes \Phi_{ij}. \end{aligned} \tag{2.27}$$

Proof. Since

$$\triangle(E_i)\triangle(F_j) = E_i F_j \otimes K_j^{-1} + E_i \otimes F_j + K_i F_j \otimes E_i K_j^{-1} + K_i \otimes E_i F_j,$$
$$\triangle(F_j)\triangle(E_i) = F_j E_i \otimes K_j^{-1} + F_j K_i \otimes K_j^{-1} E_i + E_i \otimes F_j + K_i \otimes F_j E_i,$$

and $K_i F_j \otimes E_i K_j^{-1} = F_j K_i \otimes K_j^{-1} E_i$, we get (2.25). We also get (2.26) and (2.27) by using (2.25) and (2.24). □

Recall that an antipode of a Hopf algebra is also an anti-automorphism of the corresponding coalgebra. We use above notation to prove the following result, which means that the antipode S of U_q is also an anti-automorphism of the "coalgebra" U_q.

Proposition 2.2. *It holds that*

$$\tau(S \otimes S)\triangle = \triangle S : \mathrm{U}_q \to \tilde{\otimes}^2 \mathrm{U}_q. \tag{2.28}$$

Proof. At first we show that $\tau(S \otimes S)\triangle : \mathrm{U}_q \to \tilde{\otimes}^2 \mathrm{U}_q$ preserves all defining relations of U_q. It suffices to check GIM-Serre relations. Assume that $\ell + 1 \leqslant i \neq j \leqslant \ell + 2$. Set

$$GS_1 = \sum_{s=0}^{3} (-1)^s E_i^{(3-s)} F_j E_i^{(s)}, \qquad GS_2 = E_i E_j - E_j E_i. \tag{2.29}$$

By a direct computation we have that

$$\begin{gathered} S(\Phi_{ij}) = -K_{j-i}\Phi_{ij}, \qquad S\big(\hat{T}_i(F_j)\big) = -q^{-2} K_{j-2i} \hat{T}_i'(F_j), \\ S\big(\hat{T}_i'(F_j)\big) = -q^2 K_{j-2i} \hat{T}_i(F_j). \end{gathered} \tag{2.30}$$

By these identities and (2.10) it follows that

$$\begin{aligned}\tau(S\otimes S)\triangle(GS_1) &= \frac{q^{-2}-1}{[2]}\tau(S\otimes S)(r_{ij})\\ &= \frac{1-q^2}{[2]}\big(K_{j-2i}\hat{T}_i'(F_j)\otimes K_{-3i}E_i - K_{j-i}E_i\otimes K_{j-2i}\hat{T}_i(F_j)\big)\\ &= -\frac{1-q^2}{[2]}(K_{j-3i}\otimes K_{j-3i})\big(\big(\hat{\otimes}^2\omega\big)(r_{ij})\big) = 0\in\tilde{\otimes}^2\mathrm{U}_q,\end{aligned}$$

where $\hat{\otimes}^2\omega$ is given by (2.15). Also we have that $\tau(S\otimes S)\triangle(GS_2) = 0\in\tilde{\otimes}^2\mathrm{U}_q$. Other GIM-Serre relations are similar.

It is easy to see that (2.28) holds on generators of U_q. Assume that it holds for $x, y\in\mathrm{U}_q$. We show that it also holds for xy. Let $\triangle(x) = x_1\otimes x_2\in\tilde{\otimes}^2\mathrm{U}_q$ and $\triangle(y) = y_1\otimes y_2\in\tilde{\otimes}^2\mathrm{U}_q$ (in Sweedler's notation). Then

$$\begin{aligned}\tau(S\otimes S)\triangle(xy) &= \tau(S\otimes S)(x_1y_1\otimes x_2y_2) = S(y_2)S(x_2)\otimes S(y_1)S(x_1)\\ &= \big(S(y_2)\otimes S(y_1)\big)\big(S(x_2)\otimes S(x_1)\big)\\ &= \tau(S\otimes S)\triangle(y)\tau(S\otimes S)\triangle(x)\\ &= \triangle S(y)\triangle S(x)\quad\text{(by assumption)}\\ &= \triangle\big(S(y)S(x)\big) = \triangle S(xy).\end{aligned}$$

So, $\tau(S\otimes S)\triangle$ is well defined and (2.28) holds. This completes the proof. □

One may also use (2.25) to obtain (2.10) by a little easier computation since

$$\begin{aligned}&\sum_{s=0}^{3}(-1)^s\big(\triangle(E_i)\big)^{(3-s)}\triangle(F_j)\big(\triangle(E_i)\big)^{(s)}\\ &\quad= \frac{1}{[3]}\big\{E_i^2\Phi_{ij} - \big(q^2+q^{-2}\big)E_i\Phi_{ij}E_i + \Phi_{ij}E_i^2\big\}.\end{aligned}\tag{2.31}$$

Applying $\hat{\otimes}^2\omega$ to (2.25), (2.26) and (2.27), we obtain similar identities. The fact that $1\otimes\triangle$ (respectively $\triangle\otimes 1$) is not an algebra morphism from $\tilde{\otimes}^2\mathrm{U}_q$ to $\mathrm{U}_q\otimes(\mathrm{U}_q\otimes\mathrm{U}_q)$ (respectively $(\mathrm{U}_q\otimes\mathrm{U}_q)\otimes\mathrm{U}_q$) is also illustrated by the following identities, which are direct corollaries to (2.26) and (2.27): For r_{ij} given by (2.18) and s_{ij} given by (2.19),

$$\begin{aligned}(1\otimes\triangle)(r_{ij}) &= r_{ij}\otimes K_j^{-1} + K_{2i}E_i\otimes K_{2i}\otimes\hat{T}_i(F_j) - K_i\hat{T}_i'(F_j)\otimes K_{i-j}\otimes K_j^{-1}E_i\\ &\quad+\big(q^2-q^{-2}\big)K_{2i}E_i\otimes K_i\Phi_{ij}\otimes K_j^{-1}E_i,\\ (\triangle\otimes 1)(r_{ij}) &= K_{3i}\otimes r_{ij} + K_{2i}E_i\otimes K_{2i}\otimes\hat{T}_i(F_j) - K_i\hat{T}_i'(F_j)\otimes K_{i-j}\otimes K_j^{-1}E_i\\ &\quad+\big(q^2-q^{-2}\big)K_{2i}E_i\otimes K_i\Phi_{ij}\otimes K_j^{-1}E_i,\\ (\triangle\otimes 1)(s_{ij}) &= K_{i+j}\otimes s_{ij} + K_jE_i\otimes K_j\otimes E_j - K_iE_j\otimes K_i\otimes E_i,\\ (1\otimes\triangle)(s_{ij}) &= s_{ij}\otimes 1 + K_jE_i\otimes K_j\otimes E_j - K_iE_j\otimes K_i\otimes E_i.\end{aligned}\tag{2.32}$$

These identities will be used to prove Proposition 3.3 below.

3. Coassociativity and higher order coproduct

We begin with a review on presentations of tensor product of algebras and fix some notation for our purpose. Let $\Bbbk$ be a field and A a unitary associative algebra over $\Bbbk$ with a presentation $A = \Bbbk\{X\}/I$, where X is a set and I is a two-sided ideal in the free algebra $\Bbbk\{X\}$. For any positive integer n, take n copies $X(k)$ of X and let $I(k)$ be the corresponding ideals in the free algebras $\Bbbk\{X(k)\}$ $(1 \leqslant k \leqslant n)$. Define the following quotient algebra:

$$\mathbf{T}_n(A) = \Bbbk\left\{\bigsqcup_{k=1}^{n} X(k)\right\}\Big/\mathbf{I}_n, \tag{3.1}$$

where $\bigsqcup_{k=1}^{n} X(k)$ is the disjoint union of the copies $X(k)$ of X, and the ideal $\mathbf{I}_n$ of $\Bbbk\{\bigsqcup_{k=1}^{n} X(k)\}$ is given by

$$\mathbf{I}_n = \bigl\langle I(k),\ 1 \leqslant k \leqslant n;\ x(k)x(m) - x(m)x(k) \text{ for } 1 \leqslant k < m \leqslant n\bigr\rangle,$$

where $x(k) \in X(k)$. Thus there are n ways of algebra morphism embedding ρ_n^p $(1 \leqslant p \leqslant n)$ of A into $\mathbf{T}_n(A)$ given by

$$\rho_n^p : x \mapsto x(p) \in X(p), \quad x \in X. \tag{3.2}$$

Note that the unity of $\mathbf{T}_n(A)$ is $1(1)1(2)\dots 1(n)$, where $1(i) = \rho_n^i(1)$ is the copy of the unity 1 of A. More generally, there are $\binom{k}{n}$ ways of embedding of $\mathbf{T}_k(A)$ into $\mathbf{T}_n(A)$ for $k \leqslant n$. In particular, we shall use the following embedding:

$$\rho(k)_n^{m'} : \mathbf{T}_k(A) \to \mathbf{T}_n(A) : x(j) \mapsto x(j + m' - 1) \tag{3.3}$$

for $1 \leqslant j \leqslant k$ and $1 \leqslant m' \leqslant n - (k-1)$. Note that ρ_n^p given by (3.2) is $\rho(1)_n^p$.

In the case $n = 2$, by the universal property of tensor product it follows that $A \otimes A$ is isomorphic to $\mathbf{T}_2(A)$ by $a \otimes b \mapsto \rho_2^1(a)\rho_2^b(b)$ (cf. Chapter II of [7]). Inductively, $(\dots(A \otimes A) \otimes \dots \otimes A)$ (n factors) is isomorphic to $\mathbf{T}_n(A)$. Thus, via the natural identification between any two n-fold tensor product of A with different ways of inserting parentheses, any n-tensor product of A with a fixed way of inserting parentheses is isomorphic to $\mathbf{T}_n(A)$. A typical rank one tensor in $\mathbf{T}_n(A)$ has the form $a(1)a(2)\dots a(n)$ for $a(i) \in \rho_n^i(A)$. Also, the flip map τ_2 of $A \otimes A$ induces an automorphism, denoted again by τ_2, of $\mathbf{T}_2(A)$: $\tau_2(x(1)) = x(2)$ and $\tau_2(x(2)) = x(1)$. Generally, let $\sigma \in S_n$ be the permutation such that $\sigma(12\dots n) = (n(n-1)\dots 21)$. Then it is easy to see that σ induces an automorphism τ_σ of $\mathbf{T}_n(A)$ given by

$$\tau_\sigma\bigl(x(i)\bigr) = x\bigl(\sigma(i)\bigr) \quad \text{for } x(i) \in X(i). \tag{3.4}$$

Now assume further that A is a bialgebra with a coproduct $\bar{\triangle} : A \to A \otimes A$. Thus we have an algebra morphism, denoted by $\triangle$, from A to $\mathbf{T}_2(A)$ since there is a natural isomorphism ψ from $A \otimes A$ to $\mathbf{T}_2(A)$. We have the Sweedler's notation: $\triangle(a) = a_1 a_2$ for $a \in A$, $a_1 \in \rho_2^1(A)$ and $a_2 \in \rho_2^2(A)$. Then we may interpret the algebra morphisms

$$\bar{\triangle}_n^m := 1 \otimes \dots \otimes 1 \otimes \bar{\triangle} \otimes 1 \otimes \dots \otimes 1$$

and coassociativity of $\bar{\triangle}$ as following

Lemma 3.1. *For $x \in X$, assume that $\triangle(x) = x_1 x_2 \in \mathbf{T}_2(A)$. Then, for each n and $1 \leqslant m \leqslant n$ there is a unique algebra morphism $\triangle_n^m : \mathbf{T}_n(A) \to \mathbf{T}_{n+1}(A)$ given by*

$$\triangle_n^m\big(\rho_n^k(x)\big) = \triangle_n^m\big(x(k)\big) = \begin{cases} \rho_{n+1}^{k+1}(x) & \text{for } k > m, \\ \rho_{n+1}^{k}(x_1)\rho_{n+1}^{k+1}(x_2) & \text{for } k = m, \\ \rho_{n+1}^{k}(x) & \text{for } k < m. \end{cases} \tag{3.5}$$

Moreover, $\bar{\triangle}$ is coassociative if and only if $\triangle_2^1 \circ \triangle = \triangle_2^2 \circ \triangle : A \to \mathbf{T}_3(A)$. In this case, if we define the following algebra morphisms

$$\begin{aligned} &\triangle_2 := \triangle_2^1 \circ \triangle = \triangle_2^2 \circ \triangle : A \to \mathbf{T}_2(A); \\ &\triangle_n := \triangle_n^m \circ \triangle_{n-1} : A \to \mathbf{T}_{n+1}(A), \end{aligned} \tag{3.6}$$

then $\triangle_n$ is independent of the choice of m $(1 \leqslant m \leqslant n)$.

Proof. It is direct to show that $\triangle_n^m$ given by (3.5) preserves defining relations $\mathbf{I}_n$ of $\mathbf{T}_n(A)$. By definition, $\bar{\triangle}$ is coassociative if and only if $(\bar{\triangle} \otimes 1) \circ \bar{\triangle} : A \to (A \otimes A) \otimes A$ equals to $\varphi \circ (1 \otimes \bar{\triangle}) \circ \bar{\triangle} : A \to (A \otimes A) \otimes A$, where $\varphi : A \otimes (A \otimes A) \to (A \otimes A) \otimes A$ is the canonical algebra morphism: $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$. By using the canonical isomorphism $\psi : (A \otimes A) \otimes A \to \mathbf{T}_3(A)$ and checking on generators it follows that $\psi \circ \phi \circ (1 \otimes \bar{\triangle}) \circ \bar{\triangle} = \triangle_2^2 \circ \triangle$ while $\psi((\bar{\triangle} \otimes 1) \circ \bar{\triangle}) = \triangle_2^1 \circ \triangle$. So the second statement follows. (3.6) is clear by the coassociativity. Indeed, by coassociativity of $\bar{\triangle}$ we have well-defined algebra morphisms $\bar{\triangle}_2 = (1 \otimes \bar{\triangle}) \circ \bar{\triangle} = (\bar{\triangle} \otimes 1) \circ \bar{\triangle}$ and inductively, $\bar{\triangle}_n = \bar{\triangle}_n^m \circ \bar{\triangle}_{n-1}$, which is independent of m $(1 \leqslant m \leqslant n)$. Now, for any n-fold tensor product $\otimes^n A$ with a fixed way of inserting parentheses, there is an isomorphism ψ_n from $\otimes^n A$ to $\mathbf{T}_n(A)$, which induces an isomorphism ψ_{n+1} from $\otimes^{n+1} A$, whose parentheses are determined by $\bar{\triangle}_n$, to $\mathbf{T}_{n+1}(A)$. So (3.6) follows by induction and the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\bar{\triangle}_{n-1}"] \arrow[d, equal] & \otimes^n A \arrow[r, "\bar{\triangle}_n^m"] \arrow[d, "\psi_n"'] & \otimes^{n+1} A \arrow[d, "\psi_{n+1}"'] \\ A \arrow[r, "\triangle_{n-1}"] & \mathbf{T}_n(A) \arrow[r, "\triangle_n^m"] & \mathbf{T}_{n+1}(A). \end{tikzcd}$$ □

Motivated by the above lemma, we generalize coassociativity of a coproduct as above in the following way. Assume that, for each $n \geqslant 2$, $\tilde{\otimes}^n A$ is a quotient algebra of $\mathbf{T}_n(A)$ with the canonical epimorphism $\pi_n : \mathbf{T}_n(A) \to \tilde{\otimes}^n A$. Assume that the images of generators of $\mathbf{T}_n(A)$ under π_n are non-zero. Then, any algebra morphism $\triangle : A \to \tilde{\otimes}^2 A$ still has the Sweedler's notation: $\triangle(a) = a_1 a_2$ with $a_p \in \rho_2^p(A)$, $p = 1, 2$.

Definition 3.1. An algebra morphism $\triangle : A \to \tilde{\otimes}^2 A$ is said to be *coassociative* with respect to the sequence of quotient algebras $\{\tilde{\otimes}^n A\}_{n \geqslant 2}$ if, for each $n \geqslant 2$ and $1 \leqslant m \leqslant n$, the algebra morphism $\triangle_n^m : \tilde{\otimes}^n A \to \tilde{\otimes}^{n+1} A$ induced by (3.5) satisfies the following conditions.

1. $\triangle_2^1 \circ \triangle = \triangle_2^2 \circ \triangle$.
2. Set $\triangle_1 = \triangle$, $\triangle_2 = \triangle_2^1 \circ \triangle$ and $\triangle_n = \triangle_n^m \circ \triangle_{n-1}$ for $n \geqslant 3$. Then $\triangle_n$ is independent of m $(1 \leqslant m \leqslant n)$.

For the algebra U_q given by Definition 2.1, denote by X the set of generators E_i, F_i $(1 \leqslant i \leqslant \ell+2)$, K_α $(\alpha \in \Gamma)$ and $C_j^{\pm\frac{1}{2}}$ $(j = 1, 2)$. Let I be the set of defining relations Re.1, Re.2 and Re.3. Then we have the algebra $\mathbf{T}_n(\mathrm{U}_q)$ given by (3.1) for each integer $n \geqslant 2$. In particular, the automorphism ω of U_q given by (2.14) can be extended to an automorphism, denoted again by ω, of $\mathbf{T}_n(\mathrm{U}_q)$ in the obvious way: $x(i) \mapsto (\omega(x))(i)$ for $x(i) \in X(i)$. Thus we have the following automorphism of $\mathbf{T}_n(\mathrm{U}_q)$:

$$\omega_n = \tau_\sigma \circ \omega, \tag{3.7}$$

where τ_σ is given by (3.4). Note that ω_2 coincides with $\hat{\otimes}^2\omega$ given by (2.15) modulo the canonical isomorphism between $\mathrm{U}_q \otimes \mathrm{U}_q$ and $\mathbf{T}_2(\mathrm{U}_q)$.

For $\ell+1 \leqslant i \neq j \leqslant \ell+2$, using embedding ρ_n^i of U_q in $\mathbf{T}_n(\mathrm{U}_q)$, we may rewrite elements r_{ij} and s_{ij} of $\mathrm{U}_q \otimes \mathrm{U}_q$ given by (2.18) and (2.19) respectively as following elements in $\mathbf{T}_2(\mathrm{U}_q)$:

$$R_{ij}^{(2)} = \rho_2^1(K_{2i}E_i)\rho_2^2\big(\hat{T}_i(F_j)\big) - \rho_2^1\big(K_i\hat{T}_i'(F_j)\big)\rho_2^2\big(K_j^{-1}E_i\big), \tag{3.8}$$

$$S_{ij}^{(2)} = \rho_2^1(K_jE_i)\rho_2^2(E_j) - \rho_2^1(K_iE_j)\rho_2^2(E_i). \tag{3.9}$$

Set

$$\mathbf{J}_2 := \big\langle R_{ij}^{(2)},\ S_{ij}^{(2)},\ \omega_2\big(R_{ij}^{(2)}\big),\ \omega_2\big(R_{ij}^{(2)}\big)\colon\ \ell+1 \leqslant i \neq j \leqslant \ell+2\big\rangle \tag{3.10}$$

to be the two-sided ideal in $\mathbf{T}_2(\mathrm{U}_q)$. Define the quotient algebra by

$$\pi_2 : \mathbf{T}_2(\mathrm{U}_q) \to \tilde{\otimes}^2\mathrm{U}_q := \mathbf{T}_2(\mathrm{U}_q)/\mathbf{J}_2, \tag{3.11}$$

which coincides with the quotient algebra given by (2.20), modulo the isomorphism induced by the canonical isomorphism between $\mathrm{U}_q \otimes \mathrm{U}_q$ and $\mathbf{T}_2(\mathrm{U}_q)$. Since the images of generators of $\mathbf{T}_2(\mathrm{U}_q)$ are nonzero in $\tilde{\otimes}^2\mathrm{U}_q$, for brevity, we use same symbols for generators of $\tilde{\otimes}^2\mathrm{U}_q$. Rewrite Drinfel'd–Jimbo coproduct $\triangle$ given by (2.9), denoted again by $\triangle : \mathrm{U}_q \to \tilde{\otimes}^2\mathrm{U}_q$ as following:

$$\begin{aligned}
\triangle(E_i) &= \rho_2^1(E_i)\rho_2^2(1) + \rho_2^1(K_i)\rho_2^2(E_i),\\
\triangle(F_i) &= \rho_2^1(F_i)\rho_2^2\big(K_i^{-1}\big) + \rho_2^1(1)\rho_2^2(F_i),\\
\triangle(K_\alpha) &= \rho_2^1(K_\alpha)\rho_2^2(K_\alpha),\\
\triangle\big(C_j^{\pm\frac{1}{2}}\big) &= \rho_2^1\big(C_j^{\pm\frac{1}{2}}\big)\rho_2^2\big(C_j^{\pm\frac{1}{2}}\big)
\end{aligned} \tag{3.12}$$

for $1 \leqslant i \leqslant \ell+2$, $j = 1, 2$ and $\alpha \in \Gamma$. Then Lemma 2.2 implies that $\triangle$ is an algebra morphism. Moreover, by Lemma 2.5 the following identities hold in $\mathbf{T}_2(\mathrm{U}_q)$:

$$\triangle\big(\hat{T}_i(F_j)\big)=\rho_2^1\big(\hat{T}_i(F_j)\big)\rho_2^2\big(K_j^{-1}\big)+\rho_2^1(K_{2i})\rho_2^2\big(\hat{T}_i(F_j)\big)$$
$$+\big(q^2-q^{-2}\big)\rho_2^1(K_i\Phi_{ij})\rho_2^2\big(K_j^{-1}E_i\big), \tag{3.13}$$
$$\triangle\big(\hat{T}_i'(F_j)\big)=\rho_2^1\big(\hat{T}_i'(F_j)\big)\rho_2^2\big(K_j^{-1}\big)+\rho_2^1(K_{2i})\rho_2^2\big(\hat{T}_i'(F_j)\big)$$
$$-\big(q^2-q^{-2}\big)\rho_2^1(K_iE_i)\rho_2^2(\Phi_{ij}), \tag{3.14}$$
$$\triangle(\Phi_{ij})=\rho_2^1(\Phi_{ij})\rho_2^2\big(K_j^{-1}\big)+\rho_2^1(K_i)\rho_2^2(\Phi_{ij}). \tag{3.15}$$

More generally, for $n\geqslant 3$ and $\ell+1\leqslant i\neq j\leqslant \ell+2$, we define the following elements in $\mathbf{T}_n(\mathrm{U}_q)$:

$$R_{ij}^{(n)}=\rho_n^1(K_{2i}E_i)\prod_{p=1}^{n-2}\rho_n^{p+1}(K_{2i})\rho_n^n\big(\hat{T}_i(F_j)\big)$$
$$-\rho_n^1\big(K_i\hat{T}_i'(F_j)\big)\prod_{p=1}^{n-2}\rho_n^{p+1}(K_{i-j})\rho_n^n\big(K_j^{-1}E_i\big)+\big(q^2-q^{-2}\big)\sum_{p=1}^{n-2}\rho_n^1(K_{2i}E_{2i})$$
$$\times\prod_{p'=1}^{p-1}\rho_n^{p'+1}(K_{2i})\rho_n^{p+1}(K_i\Phi_{ij})\prod_{p''=p+1}^{n-2}\rho_n^{p''+1}(K_{i-j})\rho_n^n\big(K_j^{-1}E_i\big); \tag{3.16}$$
$$S_{ij}^{(n)}=\rho_n^1(K_jE_i)\prod_{p=1}^{n-2}\rho_n^{p+1}(K_j)\rho_n^n(E_j)-\rho_n^1(K_iE_j)\prod_{p=1}^{n-2}\rho_n^{p+1}(K_i)\rho_n^n(E_i), \tag{3.17}$$

where ρ_n^p is given by (3.2) and Φ_{ij} is given by (2.23).

Let $\mathcal{S}_n$ be the subset of $\mathbf{T}_n(\mathrm{U}_q)$: For $\ell+1\leqslant i\neq j\leqslant \ell+2$,

$$\mathcal{S}_n=\big\{R_{ij}^{(n)},\ S_{ij}^{(n)},\ \rho(k)_n^{m'}\big(R_{ij}^{(k)}\big),$$
$$\rho(k)_n^{m'}\big(S_{ij}^{(k)}\big)\colon\ 1\leqslant m'\leqslant n-(k-1),\ 2\leqslant k\leqslant n-1\big\}, \tag{3.18}$$

where $\rho(k)_n^{m'}$ is given by (3.3). Let

$$\mathbf{J}_n:=\big\langle\mathcal{S}_n,\ \omega_n(\mathcal{S}_n)\big\rangle \tag{3.19}$$

be the two-sided ideal of $\mathbf{T}_n(\mathrm{U}_q)$, where ω_n is given by (3.7), and the corresponding quotient algebra is denoted by

$$\tilde{\otimes}^n\mathrm{U}_q:=\mathbf{T}_n(\mathrm{U}_q)/\mathbf{J}_n. \tag{3.20}$$

Then we have the following

Proposition 3.1. *The algebra morphism $\triangle:\mathrm{U}_q\to\tilde{\otimes}^2\mathrm{U}_q$ of Drinfel'd–Jimbo type given by* (3.12) *is coassociative in the sense of Definition* 3.1.

Proof. At first we show that $\triangle_n^m : \tilde{\otimes}^n \mathrm{U}_q \to \tilde{\otimes}^{n+1} \mathrm{U}_q$ $(1 \leqslant m \leqslant n)$ given by (3.5) is an algebra morphism by checking defining relations of $\tilde{\otimes}\mathrm{U}_q$. It is well known that Drinfel'd–Jimbo coproduct preserves defining relations Re.1 and Re.2 of U_q and hence $\triangle_n^m$ preserves the corresponding defining relations of $\mathbf{T}_n(\mathrm{U}_q)$.

Assume that $\ell+1 \leqslant i \neq j \leqslant \ell+2$. For GIM-Serre relations $GS_1, GS_2 \in I$ given by (2.29) we have the notation $GS_t(k) \in I(k)$ $(t = 1, 2)$. By (2.17), (2.18) and (3.8), it follows that

$$\triangle_n^m\big(GS_1(k)\big) = \begin{cases} \rho_{n+1}^{k+1}(GS_1) = GS_1(k+1) \in I(k+1) & \text{for } k > m, \\ \frac{q^{-2}-1}{[2]} \rho(2)_{n+1}^{k+1}(R_{ij}^{(2)}) \in \mathbf{J}_{n+1} & \text{for } k = m, \\ \rho_{n+1}^{k}(GS_1) = GS_1(k) \in I(k) & \text{for } k < m. \end{cases}$$

Similar results hold for GS_2. Applying the automorphism ω given by (2.14) of U_q, it follows that $\triangle_n^m$ preserves defining relations $I(k)$ for $k = 1, 2, \dots, n$.

Set $R_{ij}^{(n)} = A_n - B_n + C_n$, where

$$\begin{aligned} A_n &= \rho_n^1(K_{2i}E_i) \prod_{p=1}^{n-2} \rho_n^{p+1}(K_{2i}) \rho_n^n\big(\hat{T}_i(F_j)\big), \\ B_n &= \rho_n^1\big(K_i \hat{T}_i'(F_j)\big) \prod_{p=1}^{n-2} \rho_n^{p+1}(K_{i-j}) \rho_n^n\big(K_j^{-1}E_i\big), \\ C_n &= \big(q^2 - q^{-2}\big) \sum_{p=1}^{n-2} \rho_n^1(K_{2i}E_{2i}) \prod_{p'=1}^{p-1} \rho_n^{p'+1}(K_{2i}) \rho_n^{p+1}(K_i \Phi_{ij}) \\ &\quad \times \prod_{p''=p+1}^{n-2} \rho_n^{p''+1}(K_{i-j}) \rho_n^n\big(K_j^{-1}E_i\big). \end{aligned} \tag{3.21}$$

Now we compute $\triangle_n^m(R_{ij}^{(n)})$ for $1 \leqslant m \leqslant n$ by recalling the embedding $\rho(k)_n^{m'}$ (cf. (3.3)) and using (3.13), (3.14), (3.15) as follows.

(1) The case $m = 1$. Then

$$\begin{aligned} \triangle_n^1(A_n) &= A_{n+1} + \rho_{n+1}^1(K_{3i}) \rho_{n+1}^2(K_{2i}) \prod_{p=1}^{n-2} \rho_{n+1}^{p+2}(K_{2i}) \rho_{n+1}^{n+1}\big(\hat{T}_i(F_j)\big) \\ &= A_{n+1} + \rho_{n+1}^1(K_{3i}) \rho(n)_{n+1}^2(A_n); \\ \triangle_n^1(B_n) &= B_{n+1} + \rho_{n+1}^1(K_{3i}) \rho_{n+1}^2\big(K_i \hat{T}_i'(F_j)\big) \prod_{p=1}^{n-2} \rho_{n+1}^{p+2}(K_{i-j}) \rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big) \\ &\quad - \big(q^2 - q^{-2}\big) \rho_{n+1}^1(K_{2i}E_i) \rho_{n+1}^2(K_i \Phi_{ij}) \prod_{p=1}^{n-2} \rho_{n+1}^{p+2}(K_{i-j}) \rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big) \\ &= B_{n+1} + \rho_{n+1}^1(K_{3i}) \rho(n)_{n+1}^2(B_n) - \big(q^2 - q^{-2}\big) \\ &\quad \times \rho_{n+1}^1(K_{2i}E_i) \rho_{n+1}^2(K_i \Phi_{ij}) \prod_{p=1}^{n-2} \rho_{n+1}^{p+2}(K_{i-j}) \rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big); \end{aligned}$$

$$\begin{aligned}\triangle_n^1(C_n) = {} & \left(q^2-q^{-2}\right)\sum_{p=1}^{n-2}\rho_{n+1}^1(K_{2i}E_i)\rho_{n+1}^2(K_{2i})\\ & \times\prod_{p'=1}^{p-1}\rho_{n+1}^{p'+2}(K_{2i})\rho_{n+1}^{p+2}(K_i\Phi_{ij})\prod_{p''=p+1}^{n-2}\rho_{n+1}^{p''+2}(K_{i-j})\rho_{n+1}^{n+1}\left(K_j^{-1}E_i\right)\\ & + \rho_{n+1}^1(K_{3i})\rho(n)_{n+1}^2(C_n).\end{aligned}$$

Note that

$$\begin{aligned}&\left(q^2-q^{-2}\right)\rho_{n+1}^1(K_{2i}E_i)\rho_{n+1}^2(K_i\Phi_{ij})\prod_{p=1}^{n-2}\rho_{n+1}^{p+2}(K_{i-j})\rho_{n+1}^{n+1}\left(K_j^{-1}E_i\right)\\ &\quad + \left(q^2-q^{-2}\right)\sum_{p=1}^{n-2}\rho_{n+1}^1(K_{2i}E_i)\rho_{n+1}^2(K_{2i})\\ &\quad \times\prod_{p'=1}^{p-1}\rho_{n+1}^{p'+2}(K_{2i})\rho_{n+1}^{p+2}(K_i\Phi_{ij})\prod_{p''=p+1}^{n-2}\rho_{n+1}^{p''+2}(K_{i-j})\rho_{n+1}^{n+1}\left(K_j^{-1}E_i\right)\\ &= \left(q^2-q^{-2}\right)\rho_{n+1}^1(K_{2i}E_i)\sum_{p=1}^{n-1}\left(\prod_{p'=1}^{p-1}\rho_{n+1}^{p'+1}(K_{2i})\rho_{n+1}^{p+1}(K_i\Phi_{ij})\prod_{p''=p+1}^{n-1}\rho_{n+1}^{p''+1}(K_{i-j})\right)\\ &\quad \times\rho_{n+1}^{n+1}\left(K_j^{-1}E_i\right)\\ &= C_{n+1},\end{aligned}$$

it follows that

$$\triangle_n^1\left(R_{ij}^{(n)}\right) = R_{ij}^{(n+1)} + \rho_{n+1}^1(K_{3i})\rho(n)_{n+1}^2\left(R_{ij}^{(n)}\right)\in\mathbf{J}_{n+1}.$$

(2) The case $m=n$. By a completely similar computation as above we have that

$$\triangle_n^n(R_{ij}^{(n)}) = R_{ij}^{(n+1)} + \rho(n)_{n+1}^1\left(R_{ij}^{(n)}\right)\rho_{n+1}^{n+1}\left(K_j^{-1}\right)\in\mathbf{J}_{n+1}.$$

(3) The case $2\leqslant m\leqslant n-1$. It is easy to see that

$$\triangle_n^m(A_n) = A_{n+1},\qquad \triangle_n^m(B_n) = B_{n+1},$$

where A_n, B_n is given by (3.21). Since $\triangle_n^m(C_n)$ can be broken into the following three parts:

$$\begin{aligned}\triangle_n^m(C_n) = {} & \left(q^2-q^{-2}\right)\triangle_n^m\left(\sum_{p=1}^{m-2}\right) + \left(q^2-q^{-2}\right)\triangle_n^m\left(\sum_{p=m}^{n-2}\right)\\ & + \left(q^2-q^{-2}\right)\triangle_n^m\left\{\rho_n^1(K_{2i}E_i)\prod_{p'=1}^{m-2}\rho_n^{p'+1}(K_{2i})\rho_n^m(K_i\Phi_{ij})\right.\\ & \left.\times\prod_{p''=m}^{n-2}\rho_n^{p''+1}(K_{i-j})\rho_n^n\left(K_j^{-1}E_i\right)\right\},\end{aligned}$$

where

$$\begin{aligned}\triangle_n^m\left(\sum_{p=1}^{m-2}\right) &= \sum_{p=1}^{m-2}\rho_{n+1}^1(K_{2i}E_i)\prod_{p'=1}^{p-1}\rho_{n+1}^{p'+1}(K_{2i})\rho_{n+1}^{p+1}(K_i\Phi_{ij})\\ &\quad\times\prod_{p''=p+1}^{n-1}\rho_{n+1}^{p''+1}(K_{i-j})\rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big)\quad(\text{since } m>p+1),\\ \triangle_n^m\left(\sum_{p=m}^{n-2}\right) &= \sum_{p=m}^{n-2}\rho_{n+1}^1(K_{2i}E_i)\prod_{p'=1}^{p}\rho_{n+1}^{p'+1}(K_{2i})\rho_{n+1}^{p+2}(K_i\Phi_{ij})\\ &\quad\times\prod_{p''=p+1}^{n-2}\rho_{n+1}^{p''+2}(K_{i-j})\rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big)\quad(\text{since } m<p+1)\\ &= \sum_{p=m+1}^{n-2}\rho_{n+1}^1(K_{2i}E_i)\prod_{p'=1}^{p-1}\rho_{n+1}^{p'+1}(K_{2i})\\ &\quad\times\rho_{n+1}^{p+1}(K_i\Phi_{ij})\prod_{p''=p+1}^{n-1}\rho_{n+1}^{p''+1}(K_{i-j})\rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big),\end{aligned}$$

and, by (3.15),

$$\begin{aligned}&\triangle_n^m\left\{\rho_n^1(K_{2i}E_i)\prod_{p'=1}^{m-2}\rho_n^{p'+1}(K_{2i})\rho_n^m(K_i\Phi_{ij})\prod_{p''=m}^{n-2}\rho_n^{p''+1}(K_{i-j})\rho_n^n\big(K_j^{-1}E_i\big)\right\}\\ &=\rho_{n+1}^1(K_{2i}E_i)\prod_{p'=1}^{m-2}\rho_{n+1}^{p'+1}(K_{2i})\rho_{n+1}^m(K_i\Phi_{ij})\rho_{n+1}^{m+1}(K_{i-j})\prod_{p''=m}^{n-2}\rho_{n+1}^{p''+2}(K_{i-j})\rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big)\\ &\quad+\rho_{n+1}^1(K_{2i}E_i)\prod_{p'=1}^{m-2}\rho_{n+1}^{p'+1}(K_{2i})\rho_{n+1}^m(K_{2i})\rho_{n+1}^{m+1}(K_i\Phi_{ij})\prod_{p''=m}^{n-2}\rho_{n+1}^{p''+2}(K_{i-j})\rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big)\\ &=\rho_{n+1}^1(K_{2i}E_i)\prod_{p'=1}^{m-2}\rho_{n+1}^{p'+1}(K_{2i})\rho_{n+1}^m(K_i\Phi_{ij})\prod_{p''=m}^{n-1}\rho_{n+1}^{p''+1}(K_{i-j})\rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big)\\ &\quad+\rho_{n+1}^1(K_{2i}E_i)\prod_{p'=1}^{m-1}\rho_{n+1}^{p'+1}(K_{2i})\rho_{n+1}^{m+1}(K_i\Phi_{ij})\prod_{p''=m+1}^{n-1}\rho_{n+1}^{p''+2}(K_{i-j})\rho_{n+1}^{n+1}\big(K_j^{-1}E_i\big),\end{aligned}$$

summing up above three parts it follows that $\triangle_n^m(C_n)=C_{n+1}$ and hence

$$\triangle_n^m\big(R_{ij}^{(n)}\big)=R_{ij}^{(n+1)}\in\mathbf{J}_{n+1}\quad\text{for } 2\leqslant m\leqslant n-1.$$

So we have shown that $\triangle_n^m(R_{ij}^{(n)})\in\mathbf{J}_{n+1}$ for $1\leqslant m\leqslant n$. Also, it is easy to see that

$$\triangle_n^m\left(S_{ij}^{(n)}\right)=\begin{cases} S_{ij}^{(n+1)}+\rho_{n+1}^1(K_{i+j})\rho(n)_{n+1}^2(S_{ij}^{(n)}) & \text{for } m=1,\\ S_{ij}^{(n+1)} & \text{for } 2\leqslant m\leqslant n-1,\\ S_{ij}^{(n+1)}+\rho(n)_{n+1}^1(S_{ij}^{(n)})\rho_{n+1}^{n+1}(1) & \text{for } m=n,\end{cases}$$

and hence $\triangle_n^m(S_{ij}^{(n)})\in\mathbf{J}_{n+1}$ as required. At last, for $1\leqslant k,m,m'\leqslant n$, we have that

$$\triangle_n^m\left(\rho(k)_n^{m'}\left(R_{ij}^{(k)}\right)\right)=\begin{cases} \rho(k)_{n+1}^{m'+1}(R_{ij}^{(k)}) & \text{for } m-m'<0,\\ \rho(k+1)_{n+1}^{m'}(\triangle_k^{i+1}(R_{ij}^{(k)})) & \text{for } 0\leqslant m-m'\leqslant k-1,\\ \rho(k)_{n+1}^{m'}(R_{ij}^{(k)}) & \text{for } m-m'>k-1,\end{cases}$$

where the second case holds by induction on k, and hence $\triangle_n^m(\rho(k)_n^{m'}(R_{ij}^{(k)}))\in\mathbf{J}_{n+1}$ as required. By this and similar results for $\triangle_n^m(\rho(k)_n^{m'}(S_{ij}^{(k)}))$, it follows that $\triangle_n^m$ preserves all defining relations in $\mathbf{J}_n$, applying ω_n given by (3.7). So $\triangle_n^m$ $(1\leqslant m\leqslant n)$ is an algebra morphism from $\tilde{\otimes}^n\mathrm{U}_q$ to $\tilde{\otimes}^{n+1}\mathrm{U}_q$.

It is well known that the action of $\triangle_n:=\triangle_n^m\circ\triangle_{n-1}$ on generators of U_q is independent of m $(1\leqslant m\leqslant n)$. So $\triangle$ satisfies all conditions in Definition 3.1. This completes the proof. $\square$

We may go back to the usual notation on tensor products:

$$x_1\otimes x_2\otimes\cdots\otimes x_n:=\rho_1^1(x_1)\rho_2^2(x_2)\ldots\rho_n^n(x_n)\in\tilde{\otimes}^n\mathrm{U}_q\quad\text{or}\quad\mathbf{T}_n(\mathrm{U}_q)$$

for $x_i\in\mathrm{U}_q$. In particular, we still have Sweedler's notation:

$$\triangle_{n-1}(x)=x_{(1)}\otimes x_{(2)}\otimes\cdots\otimes x_{(n)}\in\tilde{\otimes}^n\mathrm{U}_q$$

for $x,x_{(p)}\in\mathrm{U}_q$. Also, the counit ε of U_q induces higher order counit. Namely, we may generalize Lemma 2.3 as following (we identify $\tilde{\otimes}^0\mathrm{U}_q$ with $\mathbb{Q}(q)$).

Proposition 3.2. *The algebra morphism $\varepsilon:\mathrm{U}_q\to\mathbb{Q}(q)$ given by (2.8) induces an algebra morphism $\varepsilon_n^p:\tilde{\otimes}^n\mathrm{U}_q\to\tilde{\otimes}^{n-1}\mathrm{U}_q$ by defining*

$$\begin{aligned}\varepsilon_n^p\left(\rho_n^1(x_1)\ldots\rho_n^n(x_n)\right)&=\varepsilon_n^p(x_1\otimes\cdots\otimes x_n)\\ &:=\varepsilon(x_p)\rho_{n-1}^1(x_1)\ldots\rho_{n-1}^{p-1}(x_{p-1})\rho_{n-1}^{p+1}(x_{p+1})\ldots\rho_{n-1}^{n-1}(x_n)\\ &=\varepsilon(x_p)x_1\otimes\cdots\otimes x_{p-1}\otimes x_{p+1}\otimes\cdots\otimes x_n\end{aligned}\tag{3.22}$$

for $x_k\in\mathrm{U}_q$ and $1\leqslant p\leqslant n$. Moreover, for any $x\in\mathrm{U}_q$ it holds that

$$\triangle_n(x)=\sum_{p=1}^{p=n+1}\varepsilon_{n+1}^p\left(\triangle_{n+1}^p(x)\right).\tag{3.23}$$

Proof. It is easy to see that

$$\varepsilon_n^p\big(R_{ij}^{(n)}\big)=\begin{cases}0 & \text{for } p=1,n,\\ R_{ij}^{(n-1)}\in \mathbf{J}_{n-1} & \text{for } 2\leqslant p\leqslant n-1,\end{cases}$$

and by induction on k,

$$\varepsilon_n^p\big(\rho(k)_n^{m'}R_{ij}^{(k)}\big)=\begin{cases}\rho(k)_{n-1}^{m'-1}(R_{ij}^{(k)}) & \text{for } p\leqslant m',\\ \rho(k-1)_n^{m'}\varepsilon_k^{p-m'}(R_{ij}^{(k)}) & \text{for } m'+1\leqslant p\leqslant m'+1+k,\\ \rho(k)_{n-1}^{m'}(R_{ij}^{(k)}) & \text{for } p>m'+1+k,\end{cases}$$

and hence $\varepsilon_n^p(\rho(k)_n^{m'}R_{ij}^{(k)})\in \mathbf{J}_{n-1}$ as required. So, ε_n^p is an algebra morphism. Now (3.23) follows since it holds on generators of U_q. □

Remark 3.1. Note that the linear map $\mathbf{m}_3:\mathbf{T}_3(\mathrm{U}_q)\to \mathrm{U}_q$ given by

$$\mathbf{m}_3\big(\rho_3^1(x)\rho_3^2(y)\rho_3^3(z)\big)=\mathbf{m}_3(x\otimes y\otimes z):=xyS(z),\quad \text{for } x,y,z\in \mathrm{U}_q,$$

does not induce a linear map from $\tilde{\otimes}^3\mathrm{U}_q$ to U_q, neither is the composition $\mathbf{m}_3\circ\triangle_2:\mathrm{U}_q\to \tilde{\otimes}^3\mathrm{U}_q\to \mathrm{U}_q$ a well-defined map. So there is no adjoint representation of U_q on itself given by the Drinfel'd–Jimbo coproduct.

However, addition to Lemma 2.4, U_q still has more general properties of Hopf algebras about multiplication, antipode and higher order coproduct such as explained in p. 79 of [13]. As an example, we give out the following

Proposition 3.3. *For any $x\in \mathrm{U}_q$ the following identities hold*:

$$\begin{aligned}
&(1)\quad \big(\mathbf{m}(1\otimes S)\otimes 1\big)(\triangle\otimes 1)\triangle(x)=1\otimes x;\\
&(2)\quad \big(\mathbf{m}(S\otimes 1)\otimes 1\big)(\triangle\otimes 1)\triangle(x)=1\otimes x;\\
&(3)\quad \big(1\otimes \mathbf{m}(1\otimes S)\big)(1\otimes\triangle)\triangle(x)=x\otimes 1;\\
&(4)\quad \big(1\otimes \mathbf{m}(S\otimes 1)\big)(1\otimes\triangle)\triangle(x)=x\otimes 1.
\end{aligned}$$

Proof. We prove (1) and other identities are similar. We show that $(\mathbf{m}(1\otimes S)\otimes 1)(\triangle\otimes 1)\triangle:\mathrm{U}_q\to\tilde{\otimes}^2\mathrm{U}_q$ is well defined. As is well known, it preserves defining relations Re.1 and Re.2 of U_q. So it suffices to check GIM-Serre relations. Assume that $\ell+1\leqslant i\neq j\leqslant \ell+2$. For GS_1 given by (2.29), by (2.17), (2.18) and (2.32), it follows that

$$\begin{aligned}
&\big(\mathbf{m}(1\otimes S)\otimes 1\big)(\triangle\otimes 1)\triangle(GS_1)\\
&\quad=\frac{q^{-2}-1}{[2]}\big(\mathbf{m}(1\otimes S)\otimes 1\big)(\triangle\otimes 1)(r_{ij})
\end{aligned}$$

$$= \frac{q^{-2}-1}{[2]}\big(\mathbf{m}(1\otimes S)\otimes 1\big)\big(K_{3i}\otimes r_{ij}+K_{2i}E_i\otimes K_{2i}\otimes \hat{T}_i(F_j) - K_i\hat{T}_i'(F_j)\otimes K_{i-j}\otimes K_j^{-1}E_i+\big(q^2-q^{-2}\big)K_{2i}E_i\otimes K_i\Phi_{ij}\otimes K_j^{-1}E_i\big).$$

By (2.30), it follows that

$$\begin{aligned}
&\big(\mathbf{m}(1\otimes S)\otimes 1\big)(K_{3i}\otimes r_{ij})\\
&\quad=\big(\mathbf{m}(1\otimes S)\otimes 1\big)\big(K_{3i}\otimes K_{2i}E_i\otimes \hat{T}_i(F_j)-K_{3i}\otimes K_i\hat{T}_i'(F_j)\otimes K_j^{-1}E_i\big)\\
&\quad=K_{3i}S(K_{2i}E_i)\otimes \hat{T}_i(F_j)-K_{3i}S\big(K_i\hat{T}_i'(F_j)\big)\otimes K_j^{-1}E_i\\
&\quad=-q^4E_i\otimes \hat{T}_i(F_j)+q^4K_j\hat{T}_i(F_j)\otimes K_j^{-1}E_i;\\
&\big(\mathbf{m}(1\otimes S)\otimes 1\big)\big(K_{2i}E_i\otimes K_{2i}\otimes \hat{T}_i(F_j)\big)=K_{2i}E_iS(K_{2i})\otimes \hat{T}_i(F_j)=q^4E_i\otimes \hat{T}_i(F_j);\\
&\big(\mathbf{m}(1\otimes S)\otimes 1\big)\big(K_i\hat{T}_i'(F_j)\otimes K_{i-j}\otimes K_j^{-1}E_i\big)=K_j\hat{T}_i'(F_j)\otimes K_j^{-1}E_i;\\
&\big(\mathbf{m}(1\otimes S)\otimes 1\big)\big(K_{2i}E_i\otimes K_i\Phi_{ij}\otimes K_j^{-1}E_i\big)\\
&\quad=K_{2i}E_iS(K_i\Phi_{ij})\otimes K_j^{-1}E_i=-q^2K_jE_i\Phi_{ij}\otimes K_j^{-1}E_i.
\end{aligned}$$

So, by (2.24), we have that

$$\begin{aligned}
&\big(\mathbf{m}(1\otimes S)\otimes 1\big)(\triangle\otimes 1)\triangle(GS_1)\\
&\quad=\frac{q^{-2}-1}{[2]}\big(q^4K_j\hat{T}_i(F_j)-K_j\hat{T}_i'(F_j)-q^2\big(q^2-q^{-2}\big)K_j\Phi_{ij}\big)\otimes K_j^{-1}E_i\\
&\quad=\frac{q^{-2}-1}{[2]}\big(K_j\big(q^4E_i\Phi_{ij}-q^2\Phi_{ij}E_i-E_i\Phi_{ij}+q^2\Phi_{ij}E_i-\big(q^4-1\big)E_i\Phi_{ij}\big)\otimes K_j^{-1}E_i\big)\\
&\quad=0
\end{aligned}$$

as required. Similarly we have that $(\mathbf{m}(1\otimes S)\otimes 1)(\triangle\otimes 1)\triangle(GS_2)=0$. Other GIM-Serre relations are similar. Note that (1) holds for generators of U_q. Assume that it holds for $x, y\in \mathrm{U}_q$. Then it suffices to show that it holds for xy. For $\triangle(x)=x_{(1)}\otimes x_{(2)}$ and $\triangle(y)=y_{(1)}\otimes y_{(2)}$, since $\triangle:\mathrm{U}_q\to \tilde{\otimes}^2\mathrm{U}_q$ is an algebra morphism, we have that

$$\begin{aligned}
&\big(\mathbf{m}(1\otimes S)\otimes 1\big)(\triangle\otimes 1)\triangle(xy)\\
&\quad=\big(\mathbf{m}(1\otimes S)\otimes 1\big)(\triangle\otimes 1)(x_{(1)}y_{(1)}\otimes x_{(2)}y_{(2)})\\
&\quad=\mathbf{m}(1\otimes S)\triangle(x_{(1)}y_{(1)})\otimes x_{(2)}y_{(2)}\\
&\quad=\big(\mathbf{m}(1\otimes S)\big(\triangle(x_{(1)})\otimes x_{(2)}\big)\big)\big(\mathbf{m}(1\otimes S)\big(\triangle(y_{(1)})\otimes y_{(2)}\big)\big)\\
&\quad=(1\otimes x)(1\otimes y)\\
&\quad=1\otimes xy
\end{aligned}$$

as required. This completes the proof. □

4. Quantum loop modules

Modules will be always assumed as left modules. Weight U_q-modules are defined in the usual way as follows. Recall notation in Section 1. We have $\Gamma = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2$ and $\alpha_{\ell+j} = \delta_j - \theta$ $(j = 1, 2)$, where θ is the highest positive root of the finite root system of $(a_{ij})_{1\leqslant i,j\leqslant \ell}$. Let ϖ_j be the fundamental weight with respect to the finite root system, that is, $\varpi_j(\alpha_k) = \delta_{jk}$ for $1 \leqslant j, k \leqslant \ell$. Extend ϖ_j by defining $\varpi_j(\delta_k) = 0$. Define ϱ_k by $\varrho_k(\alpha_i) = 0$ and $\varrho_k(\delta_j) = \delta_{kj}$ for $j, k = 1, 2$ and $1 \leqslant i \leqslant \ell$. Let $\hat{\Gamma} = \bigoplus_{i=1}^{\ell} \mathbb{Z}\varpi_i \oplus \mathbb{Z}\varrho_1 \oplus \mathbb{Z}\varrho_2$ be the weight lattice. Note that $\Gamma(\mathrm{aff}) = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \oplus \mathbb{Z}\delta_1$ and $\hat{\Gamma}(\mathrm{aff}) = \bigoplus_{i=1}^{\ell} \mathbb{Z}\varpi_i \oplus \mathbb{Z}\varrho_1$ are the corresponding objects of affine Kac–Moody algebras.

Let V be a U_q-module. For any $\lambda \in \hat{\Gamma}$ let

$$V_\lambda = \{ v \in V \mid K_\alpha v = q^{\lambda(\alpha)} v\text{: for any } z\alpha \in \Gamma \},$$

and if $V_\lambda \neq 0$ then V_λ is a weight space of V with weight λ. Let $P(V)$ be the set of weights of V. If $V = \bigoplus_{\lambda \in P(V)} V_\lambda$ then we say that V is a weight U_q-module. A trivial weight U_q-module is $\mathbb{Q}(q)$, defined by $x.1 = \varepsilon(x)$, where ε is the counit of U_q.

Let $\mathrm{U}_q(\mathrm{aff})$ be the subalgebra of $\mathrm{U}_q = \mathrm{U}_q(\mathfrak{g}_{\mathrm{gim}})$ generated by E_i, F_i $(1 \leqslant i \leqslant \ell + 1)$ and K_α $(\alpha \in \Gamma(\mathrm{aff}))$. Let $\mathrm{U}_q(\mathfrak{g})$ be the quantized universal enveloping algebra of the affine Kac–Moody algebra $\mathfrak{g}$ determined by the Cartan matrix $(a_{ij})_{1\leqslant i,j\leqslant \ell+1}$. Then, similar to the setting of quantum toroidal algebras [3], we have the following

Proposition 4.1. $\mathrm{U}_q(\mathfrak{g})$ *is isomorphic to* $\mathrm{U}_q(\mathrm{aff})$*, and there is an epimorphism from* U_q *to* $\mathrm{U}_q(\mathrm{aff})$.

Proof. Let e_i, f_i $(1 \leqslant i \leqslant \ell + 1)$ and k_α $(\alpha \in \Gamma(\mathrm{aff}))$ be Chevalley generators of $\mathrm{U}_q(\mathfrak{g})$. By Drinfel'd–Jimbo's definition of $\mathrm{U}_q(\mathfrak{g})$, there is an algebra morphism $\jmath : \mathrm{U}_q(\mathfrak{g}) \to \mathrm{U}_q$ given by $\jmath(e_i) = E_i$, $\jmath(f_i) = F_i$ and $\jmath(k_\alpha) = K_\alpha$. On the other hand, for $1 \leqslant i \leqslant \ell + 2$, in U_q it holds that

$$\begin{aligned} &E_i^3 F_i - [3]E_i^2 F_i E_i + [3]E_i F_i E_i^2 - F_i E_i^3 = 0, \\ &F_i^3 E_i - [3]F_i^2 E_i F_i + [3]F_i E_i F_i^2 - E_i F_i^3 = 0. \end{aligned} \tag{4.1}$$

Indeed, since $E_i F_i - F_i E_i = Q_i := \frac{K_i - K_i^{-1}}{q - q^{-1}}$ and the left-hand side of the first formula in (4.1) is

$$\begin{aligned} &E_i^2 Q_i - (q^2 + q^{-2}) E_i Q_i E_i + Q_i E_i^2 \\ &\quad = E_i^2 \frac{K_i - K_i^{-1} - (q^2 + q^{-2})(q^2 K_i - q^{-2} K_i^{-1}) + q^4 K_i - q^{-4} K_i^{-1}}{q - q^{-1}} = 0 \end{aligned}$$

as required. The second formula in (4.1) is obtained by applying ω of U_q. It follows that there is an epimorphism $\pi : \mathrm{U}_q \to \mathrm{U}_q(\mathfrak{g})$ given by $\pi(E_i) = e_i$, $\pi(F_i) = f_i$ $(1 \leqslant i \leqslant \ell + 1)$, $\pi(K_\alpha) = k_\alpha$ $(\alpha \in \Gamma(\mathrm{aff}))$, and $\pi(E_{\ell+2}) = e_{\ell+1}$, $\pi(F_{\ell+2}) = f_{\ell+2}$, $\pi(K_{\alpha_{\ell+2}}) = k_{\alpha_{\ell+1}}$. Indeed, clearly (4.1) holds for generators of $\mathrm{U}_q(\mathfrak{g})$ and hence π preserves the GIM-Serre relations (2.4) and (2.5). Since the composition map $\pi \circ \jmath$ is the identity on $\mathrm{U}_q(\mathfrak{g})$ it follows that $\jmath$ is injective and $\mathrm{U}_q(\mathfrak{g}) \simeq \mathrm{U}_q(\mathrm{aff})$ as required. $\square$

Thus any $\mathrm{U}_q(\mathrm{aff})$-module becomes a U_q-module, whose module structure is induced by the above epimorphism from U_q to $\mathrm{U}_q(\mathrm{aff})$. More explicitly, for any $\mathrm{U}_q(\mathrm{aff})$-module V, the induced U_q-module structure is given by $E_{\ell+2}.v = E_{\ell+1}.v$, $F_{\ell+2}.v = F_{\ell+1}.v$ and $K_{\ell+2}.v = K_{\ell+1}.v$ for any $v \in V$. Denote this U_q-module by $\tilde{V}$. Clearly, if V is an irreducible (respectively weight) $\mathrm{U}_q(\mathrm{aff})$-module, then $\tilde{V}$ is also an irreducible (respectively weight) U_q-module.

Let $V_1, \ldots, V_n$ be $\mathrm{U}_q(\mathrm{aff})$-modules. Since $\mathrm{U}_q(\mathrm{aff}) \simeq \mathrm{U}_q(\mathfrak{g})$ is a Hopf algebra, $V_1 \otimes \cdots \otimes V_n$ (with any fixed way of inserting parentheses) is a tensor module of $\mathrm{U}_q(\mathrm{aff})$, whose module structure is induced by the usual Drinfel'd–Jimbo coproduct. Thus $V_1 \otimes \cdots \otimes V_n$ turns into a U_q-module via the above epimorphism from U_q to $\mathrm{U}_q(\mathrm{aff})$. However, this module needs not to be a tensor module induced by the coproduct of U_q. If $V_1, \ldots, V_n$ are weight $\mathrm{U}_q(\mathrm{aff})$-modules, we shall show that $V_1 \otimes \cdots \otimes V_n$ is a tensor U_q-module induced by the coproduct. In fact, it suffices to show that $V_1 \otimes \cdots \otimes V_n$ is also a $\tilde{\otimes}^n \mathrm{U}_q$-module. Moreover, during our computation, we find this can be generalized to quantum loop modules, which also help to construct new U_q-modules from $\mathrm{U}_q(\mathrm{aff})$-modules.

To precede we give out at first the following

Lemma 4.1. *Let V be a $\mathrm{U}_q(\mathrm{aff}) \simeq \mathrm{U}_q(\mathfrak{g})$-module (not necessarily a weight module). Fix $\ell + 1 \leqslant i \neq j \leqslant \ell + 2$. Then, for any $v \in \tilde{V}$ it holds that*

$$\begin{aligned}
&(1) \quad \Phi_{ij}.v = \Phi_{ji}.v = Q_i.v := \frac{K_i - K_i^{-1}}{q - q^{-1}}.v,\\
&(2) \quad \hat{T}_i(F_j).v = \hat{T}_j(F_i).v = -[2]K_i^{-1}E_i.v,\\
&(3) \quad \hat{T}_i'(F_j).v = \hat{T}_j'(F_i) = -[2]K_iE_i.v,
\end{aligned}$$

where Φ_{ij} is given by (2.23), *$\hat{T}_i(F_j)$ and $\hat{T}_i'(F_j)$ are given by* (2.16).

Proof. It follows directly by definition. □

Stimulated by Chari–Greenstein's definition of quantum loop modules [2], consider the algebra $\mathbb{Q}(q)[t, t^{-1}]$ over $\mathbb{Q}(q)$. For any $\mathrm{U}_q(\mathrm{aff})$-module V, define

$$L(V) = V \otimes_{\mathbb{Q}(q)} \mathbb{Q}(q)\big[t, t^{-1}\big]. \tag{4.2}$$

Note that U_q is $\mathbb{Z}$-graded by setting $|E_{\ell+2}| = 1$, $|F_{\ell+2}| = -1$ and $|x| = 0$ for other generators of U_q. Thus $\mathrm{U}_q = \bigoplus_{r\in\mathbb{Z}} (\mathrm{U}_q)_r$, where $(\mathrm{U}_q)_r$ is the subspace of elements of degree r. Then we have the following

Lemma 4.2. *For any $\mathrm{U}_q(\mathrm{aff})$-module V, $L(V)$ becomes a U_q-module by*

$$x.\big(v \otimes t^s\big) = x.v \otimes t^{s+|x|} \tag{4.3}$$

for any homogeneous element $x \in \mathrm{U}_q$ and $v \in V$, where the action $x.v$ is determined by the above U_q-module structure on $\tilde{V}$.

Proof. For $i=\ell+1$, $j=\ell+2$, by definition and (4.1), it follows that

$$\begin{aligned}&\big(E_i^3F_j-[3]E_i^2F_jE_i+[3]E_iF_jE_i^2-F_jE_i^3\big).\big(v\otimes t^s\big)\\&\quad=\big(E_i^3F_i-[3]E_i^2F_iE_i+[3]E_iF_iE_i^2-F_iE_i^3\big).v\otimes t^{s-1}=0.\end{aligned}$$

Similarly,

$$\begin{aligned}&\big(E_j^3F_i-[3]E_j^2F_iE_j+[3]E_jF_iE_j^2-F_iE_j^3\big).\big(v\otimes t^s\big)\\&\quad=\big(E_i^3F_i-[3]E_i^2F_iE_i+[3]E_iF_iE_i^2-F_iE_i^3\big).v\otimes t^{s+3}=0.\end{aligned}$$

Other defining relations of U_q are similar. □

Remark 4.1. The underlying space of the U_q-module $L(V)$ given by (4.2) is the same as that of the so-called *quantum loop module* associated to the $\mathrm{U}_q(\mathrm{aff})\simeq\mathrm{U}_q(\mathfrak{g})$-module V in [2], where the $\mathrm{U}_q(\mathfrak{g})$-module structure is extended from V via a gradation on $\mathrm{U}_q(\mathfrak{g})$ induced by the derivative. Quantum loop modules in [2] are used to study simple integrable $\mathrm{U}_q(\mathfrak{g})$-modules. In our setting we extend the $\mathrm{U}_q(\mathfrak{g})$-module structure to U_q via a gradation induced by the extra generators, that is, by defining the action of generators $E_{\ell+2}$ and $F_{\ell+2}$ on the parameter t. Our computation shows that more parameters would be introduced to construct quantum loop modules of [2] when derivatives of U_q are involved for further study of infinite-dimensional U_q-modules. Since the aim in the present paper of introducing $L(V)$ is to construct infinite-dimensional U_q-modules which are tensor modules, the action of derivatives is not necessary at the moment. It is expected Chari–Greenstein's construction would be very helpful to study simple U_q-modules.

Remark 4.2. If V is a simple $\mathrm{U}_q(\mathrm{aff})$-module which is integrable with highest weight, then $L(V)$ is also a simple U_q-module with highest weight. Indeed, $L(V)$ is a graded U_q-module with respect the above $\mathbb{Z}$-gradation: $(\mathrm{U}_q)_r(V\otimes t^s)\subseteq V\otimes t^{r+s}$. For any submodule S of $L(V)$, since V is a weight $\mathrm{U}_q(\mathfrak{g})$-module, it follows that $S=L(S_1)$ for some submodule S_1 of V.

We have the following

Proposition 4.2. *Assume that $V_1,\dots,V_n$ $(n\geqslant 2)$ are weight $\mathrm{U}_q(\mathrm{aff})$-modules. Then $L(V_1\otimes\cdots\otimes V_n)$ is a $\tilde{\otimes}^n\mathrm{U}_q$-module. Thus $L(V_1\otimes\cdots\otimes V_n)$ is a tensor U_q-module induced by the coproduct $\triangle_{n-1}:\mathrm{U}_q\to\tilde{\otimes}^n\mathrm{U}_q$. The same holds for $V_1\otimes\cdots\otimes V_n$.*

Proof. By the lemma as above, the $\mathbf{T}_n(\mathrm{U}_q)$-module structure on $L(V_1\otimes\cdots\otimes V_n)$ is given by

$$\begin{aligned}&\rho_n^i(x).\big((v_1\otimes\cdots\otimes v_n)\otimes t^s\big)\\&\quad=(v_1\otimes\cdots\otimes v_{i-1}\otimes x.v_i\otimes v_{i+1}\otimes\cdots\otimes v_n)\otimes t^{s+|x|},\end{aligned}$$

for homogeneous $x\in\mathrm{U}_q$ and the action $x.v_i$ is determined by $\tilde{V}_i$. It suffices to show that

$$\mathbf{J}_n.L(V_1\otimes\cdots\otimes V_n)=0,$$

where $\mathbf{J}_n$ is given by (3.19). Assume that $i=\ell+1,\ j=\ell+2$. The case in which $i=\ell+2,\ j=\ell+1$ is similar. Fix any weight vector v_p of V_p. Then $K_{i-j}.v_p=v_p$. Then, by applying ω_n of $\mathbf{T}_n(\mathrm{U}_q)$, it suffices to show that, for any $2\leqslant k\leqslant n$ and $1\leqslant m'\leqslant n-k+1$,

$$\rho(k)_n^{m'}\big(S_{ij}^{(k)}\big).\big((v_1\otimes\cdots\otimes v_n)\otimes t^s\big)=0, \tag{4.4}$$

$$\rho(k)_n^{m'}\big(R_{ij}^{(k)}\big).\big((v_1\otimes\cdots\otimes v_n)\otimes t^s\big)=0. \tag{4.5}$$

Since $K_i v_p=K_j v_p$ for weight vectors, (4.4) is clear by definition. To show (4.5), without loss of generality, we may assume that $m'=1$.

By Lemma 4.1, it follows that

$$\begin{aligned}
\rho(2)_n^1\big(R_{ij}^{(2)}\big).\big((v_1\otimes\cdots\otimes v_n)\otimes t^s\big) &= \big(K_{2i}E_i.v_1\otimes \hat{T}_i(F_j).v_2\otimes v_3\otimes\cdots\otimes v_n\big)\otimes t^{s-1}\\
&\quad -\big(K_i\hat{T}_i'(F_j)v_1\otimes K_j^{-1}E_i.v_2\otimes v_3\otimes\cdots\otimes v_n\big)\otimes t^{s-1}\\
&= -[2]\big(K_{2i}E_i.v_1\otimes K_i^{-1}E_i.v_2\otimes v_3\otimes\cdots\otimes v_n\big)\otimes t^{s-1}\\
&\quad +[2]\big(K_{2i}E_i.v_1\otimes K_i^{-1}E_i.v_2\otimes v_3\otimes\cdots\otimes v_n\big)\otimes t^{s-1}\\
&=0
\end{aligned}$$

as required. Now assume that $3\leqslant k\leqslant n$. By definition of $R_{ij}^{(k)}$ (cf. (3.16)), we write

$$\rho(k)_n^1\big(R_{ij}^{(k)}\big).\big((v_1\otimes\cdots\otimes v_n)\otimes t^s\big)=A-B+\sum_{p=1}^{n-2}C_p,$$

where A, B, C_p are computed as follows. At first, note that, by Lemma 4.1,

$$\begin{aligned}
A &:= \rho(k)_n^1\big(\rho_k^1(K_{2i}E_i)\rho_k^2(K_{2i})\ldots\rho_k^{k-1}(K_{2i})\rho_k^k\big(\hat{T}_i(F_j)\big)\big).\big((v_1\otimes\cdots\otimes v_n)\otimes t^s\big)\\
&= -[2]\big(K_{2i}E_i.v_1\otimes K_{2i}.v_2\otimes\cdots\otimes K_{2i}.v_{k-1}\otimes K_i^{-1}E_i.v_k\otimes v_{k+1}\otimes\cdots\otimes v_n\big)\otimes t^{s-1};\\
B &:= \rho(k)_n^1\big(\rho_k^1\big(K_i\hat{T}_i'(F_j)\big)\rho_k^2(K_{i-j})\ldots\rho_k^{k-1}(K_{i-j})\rho_k^k\big(K_j^{-1}E_i\big)\big).\big((v_1\otimes\cdots\otimes v_n)\otimes t^s\big)\\
&= -[2]\big(K_{2i}E_i.v_1\otimes K_{i-j}.v_2\otimes\cdots\otimes K_{i-j}.v_{k-1}\otimes K_i^{-1}E_i.v_k\otimes v_{k+1}\otimes\cdots\otimes v_n\big)\otimes t^{s-1}\\
&= -[2]\big(K_{2i}E_i.v_1\otimes v_2\otimes\cdots\otimes v_{k-1}\otimes K_i^{-1}E_i.v_k\otimes v_{k+1}\otimes\cdots\otimes v_n\big)\otimes t^{s-1}.
\end{aligned}$$

For each $1\leqslant p\leqslant k-2$, since

$$K_iQ_iv_{p+1}=\frac{K_{2i}-1}{q-q^{-1}}v_{p+1},$$

by Lemma 4.1, it follows that

$$\begin{aligned}
C_p &:= \left(q^2 - q^{-2}\right)\left(\rho(k)_n^1\left(\rho_k^1(K_{2i}E_i)\rho_k^2(K_{2i})\dots\rho_k^p(K_{2i})\right.\right.\\
&\qquad \left.\left.\times\, \rho_k^{p+1}(K_i\Phi_{ij})\rho_k^{p+2}(K_{i-j})\dots\rho_k^{k-1}(K_{i-j})\rho_k^k\left(K_j^{-1}E_i\right)\right)\right).\left((v_1\otimes\cdots\otimes v_n)\otimes t^s\right)\\
&= [2]\left(K_{2i}E_i.v_1\otimes K_{2i}.v_2\otimes\cdots\otimes K_{2i}.v_p\otimes K_{2i}.v_{p+1}\right.\\
&\qquad \left.\otimes\, v_{p+2}\otimes\cdots\otimes v_{k-1}\otimes K_i^{-1}E_i.v_k\otimes v_{k+1}\otimes\cdots\otimes v_n\right)\otimes t^{s-1}\\
&\qquad - [2]\left(K_{2i}E_i.v_1\otimes K_{2i}v_2\otimes\cdots\otimes K_{2i}.v_p\otimes v_{p+1}\right.\\
&\qquad \left.\otimes\, v_{p+2}\otimes\cdots\otimes v_{k-1}\otimes K_i^{-1}E_i.v_k\otimes v_{k+1}\otimes\cdots\otimes v_n\right)\otimes t^{s-1}.
\end{aligned}$$

Summing up, we get

$$\rho(k)_n^1\left(R_{ij}^{(k)}\right).\left((v_1\otimes\cdots\otimes v_n)\otimes t^s\right) = A - B + \sum_{p=1}^{n-2} C_p = 0$$

as required. This completes the proof. $\square$

However, $L(V_1)\otimes\cdots\otimes L(V_n)$ needs not to be a $\tilde{\otimes}^n\mathrm{U}_q$-module, even for weight $\mathrm{U}_q(\mathrm{aff})$-modules. So, to describe the subcategory of U_q-modules satisfying fusion rule (that is, the tensor product of U_q-modules is again a U_q-module) remains unclear for us. At least, all weight modules of the corresponding quantized affine Kac–Moody algebra belong to such a subcategory.

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